# Mass Purges <br> Supplemental Appendix 

In all that follows, since second-period actions are subsumed in $V_{2}(\tau)$ (for a subordinate) and $W_{2}(\tau)$ for the autocrat, we ignore the first-period time subscript in all the proofs.

## A Proofs for Section 3

## Proof of Lemma 1

First, observe that due to our equilibrium refinement, there is no equilibrium in which agents exert zero effort. If there is, a successful project is an out-of-equilibrium event. As it is treated as a mistake, it does not affect the autocrat's purging decision. Hence, congruent types have a profitable deviation to exert effort. The rest of the proof proceeds by contradiction.

Anticipating the proof of Lemma 2, an agent's effort as a function of his type is $e^{i}(\tau)=\max \{(1-$ $\left.\left.\kappa_{S}\right) v(S, \tau)+\left(\kappa_{F}-\kappa_{S}\right)\left(V_{2}(\tau)+L\right), 0\right\}$.

First, suppose there is an equilibrium in which the autocrat purges a greater proportion of successful agents (i.e., $\kappa_{F}>0$ and $\kappa_{S}>\kappa_{F}$ ). Then both types exert no effort, contradicting the first paragraph. Second, suppose that there is an equilibrium in which $0<\kappa_{S}<\kappa_{F}<1$. The autocrat must then be indifferent between purging from the success and failure pools. Congruent agents, however, exert more effort than non-congruent subordinates and the autocrat's posterior after success is strictly higher than her posterior after observing failure. Hence, the autocrat is never indifferent, a contradiction.

## Proof of Lemma 2

Taking $\kappa_{L}$ and $\kappa_{S}$ as given, the maximization problem of a type $\tau \in\{c, n c\}$ agent assumes the following form:

$$
\begin{equation*}
\max _{e \in[0,1]} R+e\left[\left(1-\kappa_{S}\right)\left(v(S, \tau)+V_{2}(\tau)\right)+\kappa_{S}(-L)\right]+(1-e)\left[\left(1-\kappa_{F}\right)\left(V_{2}(\tau)\right)+\kappa_{F}(-L)\right]-\frac{e^{2}}{2} \tag{A.1}
\end{equation*}
$$

If his project is successful (probability $e$ ), an agent survives the purge with probability $1-\kappa_{S}$ and receives a flow payoff $v(S, \tau)$ and as well as his second period expected payoff. If his project fails, he survives the purge with probability $1-\kappa_{F}$ and only receives his second period expected payoff then. When the agent is purged, he suffers a loss $L$.
Taking the first-order condition, we obtain:

$$
e^{i}(\tau)=\left(1-\kappa_{S}\right) v(S, \tau)+\left(\kappa_{F}-\kappa_{S}\right)\left(V_{2}(\tau)+L\right)
$$

Using Lemma 1 yields the claim.

Observe that if $-R<v(S, n c)<0$, there exists $\kappa_{F}^{0}(L)=\frac{-v(S, n c)}{R+L} \in(0,1)$ such that for all $\kappa_{F} \leq \kappa_{F}^{0}$, a non-congruent subordinate exerts zero effort in period 1 . It can be checked that for all $\kappa_{F} \in\left[0, \kappa_{F}^{0}(L)\right)$, the autocrat's posterior $\mu^{F}$ is strictly decreasing with $\kappa_{F}$ (see the proof of Lemma A.1) and $L$ (see the proof of Lemma B.1) as only congruent types exit the failure pool. As a result, all the comparative statics with respect to violence we establish in the main text (purge inference, purge breadth, effort, selection) hold for all $L$ such that the equilibrium inference $\kappa_{F}^{*}(L)$ satisfies $\kappa_{F}^{*}(L) \in\left[0, \kappa_{F}^{0}(L)\right)$. Hence, assuming $v(S, n c) \geq 0$ is without loss of generality.

In all that follows, we slightly abuse notation and denote $e^{i}\left(\kappa_{F}, L ; \tau\right)$ a type $\tau \in\{c, n c\}$ agent's effort in a partially discriminate purge and $e^{i}\left(\kappa_{S}, L ; \tau\right)$ his effort in a semi-indiscriminate purge as a function of the purge incidence and violence. Similarly, average effort is denoted by $\bar{e}\left(\kappa_{\omega}, L\right), \omega \in$ $\{F, S\}$.

Further, slightly abusing notation and given that the autocrat correctly anticipates agents' effort in equilibrium, we denote $\mu^{F}\left(\kappa_{F}, L\right)$ the autocrat's posterior that an agent is congruent conditional on failure in a partially discriminate purge $\mu^{S}\left(\kappa_{S}, L\right)$ the same posterior conditional on success in a semi-indiscriminate purge.
The next Lemma characterizes some properties of the autocrat's posteriors treating purge inferences as exogenous.

Lemma A.1. (i) In a partially discriminate purge, the autocrat's posterior after failure is strictly decreasing and concave in $\kappa_{F}$.
(ii) In a semi-indiscriminate purge, the autocrat's posterior after success is constant in $\kappa_{S}$.

Proof. We prove the lemma using a slightly general reasoning to illustrate that our results do not depend on our functional form assumptions.

Point (i). By Bayes' Rule, $\mu^{F}\left(\kappa_{F}, L\right)=\lambda \frac{1-e^{i}\left(\kappa_{F}, L ; c\right)}{1-\bar{e}\left(\kappa_{F}, L\right)}$. The relevant comparative statics is then (omitting superscript):

$$
\begin{align*}
\frac{\partial \mu^{F}\left(\kappa_{F}, L\right)}{\partial \kappa_{F}}= & \frac{\lambda}{\left(1-\bar{e}\left(\kappa_{F}, L\right)\right)^{2}}\left[-\frac{e^{i}\left(\kappa_{F}, L ; c\right)}{\partial \kappa_{F}}\left(1-\lambda e^{i}\left(\kappa_{F}, L ; c\right)-(1-\lambda) e^{i}\left(\kappa_{F}, L ; n c\right)\right)\right. \\
& \left.+\left(1-e^{i}\left(\kappa_{F}, L ; c\right)\right)\left(\lambda \frac{\partial e^{i}\left(\kappa_{F}, L ; c\right)}{\partial \kappa_{F}}+(1-\lambda) \frac{\partial e^{i}\left(\kappa_{F}, L ; n c\right)}{\partial \kappa_{F}}\right)\right] \\
= & \frac{\lambda(1-\lambda)\left(1-e^{i}\left(\kappa_{F}, L ; c\right)\right)\left(1-e^{i}\left(\kappa_{F}, L ; n c\right)\right)}{\left(1-\bar{e}\left(\kappa_{F}, L\right)\right)^{2}}\left[\frac{\frac{\partial e^{i}\left(\kappa_{F}, L ; n c\right)}{\partial \kappa_{F}}}{1-e^{i}\left(\kappa_{F}, L ; n c\right)}-\frac{\frac{\partial e^{i}\left(\kappa_{F}, L ; c\right)}{\partial \kappa_{F}}}{1-e^{i}\left(\kappa_{F}, L ; c\right)}\right] \tag{A.2}
\end{align*}
$$

By examination of Equation 5. $e^{i}\left(\kappa_{F}, L ; c\right)>e^{i}(\kappa, F ; n c)$ and $\frac{\partial e^{i}\left(\kappa_{F}, L ; c\right)}{\partial \kappa_{F}}>\frac{\partial e^{i}\left(\kappa_{F}, L ; n c\right)}{\partial \kappa_{F}}$. This directly implies: $\frac{\frac{\partial e^{i}\left(\kappa_{F}, L ; n c\right)}{\partial \kappa_{F}}}{1-e^{i}\left(\kappa_{F}, L ; n c\right)}-\frac{\frac{\partial e^{2}\left(\kappa_{F}, L ; c\right)}{\partial k_{F}}}{1-e^{i}\left(\kappa_{F}, L ; c\right)}<0$ and $\frac{\partial \mu^{F}\left(\kappa_{F}, L\right)}{\partial \kappa_{F}}<0$ as claimed.
To see that the posterior is strictly concave in $\kappa_{F}$, notice that:

$$
\begin{align*}
\frac{\left.\partial^{2} \mu^{F}\left(\kappa_{F}, L\right)\right)}{\partial \kappa_{F}^{2}} \propto & {\left[\frac{\partial^{2} e^{i}\left(\kappa_{F}, L ; n c\right)}{\partial \kappa_{F}^{2}}\left(1-e^{i}\left(\kappa_{F}, L ; c\right)\right)-\frac{\partial^{2} e^{i}\left(\kappa_{F}, L ; c\right)}{\partial \kappa_{F}^{2}}\left(1-e^{i}\left(\kappa_{F}, L ; n c\right)\right)\right]\left(1-\bar{e}\left(\kappa_{F}, L\right)\right) } \\
& +2 \frac{\partial \bar{e}\left(\kappa_{F}, L\right)}{\partial \kappa_{F}}\left[\frac{\partial e^{i}\left(\kappa_{F}, L ; n c\right)}{\partial \kappa_{F}}\left(1-e^{i}\left(\kappa_{F}, L ; c\right)\right)-\frac{\partial e^{i}\left(\kappa_{F}, L ; c\right)}{\partial \kappa_{F}}\left(1-e^{i}\left(\kappa_{F}, L ; n c\right)\right)\right] \tag{A.3}
\end{align*}
$$

Equation 5yields that $\frac{\partial^{2} e^{i}\left(\kappa_{F}, L ; \tau\right)}{\partial \kappa_{F}^{2}}=0, \tau \in\{c, n c\}$. Further, the term on the second line is negative by Equation A. 2 and $\frac{\partial \bar{e}\left(\kappa_{F}, L\right)}{\partial \kappa_{F}}>0$.
Point (ii). By Bayes' rule, the autocrat's posterior after success is: $\mu^{S}\left(\kappa_{S}, L\right)=\lambda \frac{e^{i}\left(\kappa_{S}, L ; c\right)}{\bar{e}\left(\kappa_{S}, L\right)}$. By a similar reasoning as above, we obtain:

$$
\begin{align*}
\frac{\partial \mu^{S}\left(\kappa_{S}, L\right)}{\partial \kappa_{S}}= & \frac{\lambda}{\bar{e}\left(\kappa_{S}, L\right)^{2}}\left[\frac{e^{i}\left(\kappa_{S}, L ; c\right)}{\partial \kappa_{S}}\left(\lambda e^{i}\left(\kappa_{S}, L ; c\right)+(1-\lambda) e^{i}\left(\kappa_{S}, L ; n c\right)\right)\right. \\
& \left.-e^{i}\left(\kappa_{S}, L ; c\right)\left(\lambda \frac{\partial e^{i}\left(\kappa_{S}, L ; c\right)}{\partial \kappa_{S}}+(1-\lambda) \frac{\partial e^{i}\left(\kappa_{S}, L ; n c\right)}{\partial \kappa_{S}}\right)\right] \\
= & \frac{\lambda(1-\lambda) e^{i}\left(\kappa_{S}, L ; c\right) e^{i}\left(\kappa_{S}, L ; n c\right)}{\bar{e}\left(\kappa_{S}, L\right)^{2}}\left[\frac{\frac{\partial e^{i}\left(\kappa_{S}, L ; c\right)}{\partial \kappa_{S}}}{e^{i}\left(\kappa_{S}, L ; c\right)}-\frac{\frac{\partial e^{i}\left(\kappa_{S}, L ; n c\right)}{\partial \kappa_{S}}}{e^{i}\left(\kappa_{S}, L ; n c\right)}\right] \tag{A.4}
\end{align*}
$$

Using Equation 5. we obtain: $\frac{\partial e^{i}\left(\kappa_{S}, L ; \tau\right)}{e^{i}\left(\kappa_{S}, L ; \tau\right)}=\frac{1}{1-\kappa_{S}}, \tau \in\{c, n c\}$. So $\frac{\partial \mu^{S}\left(\kappa_{S}, L\right)}{\partial \kappa_{S}}=0$ as claimed.

## Proof of Remark 1

Direct from Lemma A. 1 .

## Proof of Lemma 3

Point (i). Consider the function

$$
\begin{equation*}
K_{P D}\left(\kappa_{F}, L\right)=\beta\left(r-\mu^{F}\left(\kappa_{F}, L\right)\right) \mathcal{D}_{2}^{c, n c}-\left(C_{0}+C_{1} \kappa_{F}\left(1-\left(\bar{v}+\kappa_{F}\left(\overline{V_{2}}+L\right)\right)\right)\right) \tag{A.5}
\end{equation*}
$$

Observe that $K_{P D}\left(\kappa_{F}, L\right)$ is strictly convex in $\kappa_{F}$ (using Lemma A.1). Since all agents and the principal correctly anticipate each other actions, the equilibrium purge incidence if interior must be a solution to $K_{P D}\left(\kappa_{F}, L\right)=0$. Under the condition of the Lemma, $K_{P D}(1, L)<0$. Given the properties of $K_{P D}(\cdot, L), K_{P D}\left(\kappa_{F}, L\right)$ crosses 0 either once (from above) or zero ${ }^{1}$ The equilibrium purge incidence is thus unique and equals $\kappa_{F}^{*}(L)=0$ if $C_{0}>\left(r-\lambda \frac{1-v(S ; c)}{1-\bar{v}}\right) \mathcal{D}_{2}^{c, n c}$ or the unique solution to $K_{P D}\left(\kappa_{F}^{*}(L), L\right)=0$ otherwise.

Point (ii). Under the condition of point (ii), we have three possibilities (a) $K_{P D}\left(\kappa_{F}, L\right) \geq 0$ for all $\kappa_{F} \in[0,1]$, (b) there exists a unique solution to $K_{P D}\left(\kappa_{F}, L\right)=0$ (with $K_{P D}\left(\kappa_{F}, L\right)$ crossing 0 from below), (c) there exists two solutions to $K_{P D}\left(\kappa_{F}, L\right)=0$. In case (a), the unique equilibrium purge incidence is $\kappa_{F}^{*}(L)=1$. In cases (b) and (c), denote $\kappa_{F}^{\prime} \in(0,1)$ an interior solution (unique or not) and $\kappa_{F}^{c}=1$ the corner solution. Since our equilibrium selection selects the purge with the largest purge inference, the equilibrium must then satisfy $\kappa_{F}^{*}(L)=1$ as claimed.

Point (iii). Consider the function:

$$
\begin{equation*}
\left.K_{S D}\left(\kappa_{S}, L\right)=\beta\left(r-\mu^{S}\left(\kappa_{S}, L\right)\right) \mathcal{D}_{2}^{c, n c}-\left(C_{0}+C_{1}\left(1-\left(1-\kappa_{S}\right)^{2}\left(\bar{v}+\overline{V_{2}}+L\right)\right)\right)\right) \tag{A.6}
\end{equation*}
$$

Using Lemma A.1, $K_{S D}\left(\kappa_{S}, L\right)$ is decreasing and convex in $\kappa_{S}$. Under the condition of the point (iii) and assumption that $\beta r \mathcal{D}^{c, n c}<C_{0}+C_{1}$, we further obtain $K_{S D}(0, L)>0$ and $\lim _{\kappa_{S} \rightarrow 1} K_{S D}\left(\kappa_{S}, L\right)<0$. This implies that there exists a unique interior solution to $K_{S D}\left(\kappa_{S}, L\right)=0$ and thus a unique equilibrium purge incidence $\kappa_{S}^{*}(L)$.

As the proof of Lemma 3 highlights, our equilibrium criterion plays a role only when the conditions of point (ii) of the Lemma are satisfied. Alternative criterion selection might select different purge inference. For example, it can be checked that the equilibrium criterion based on the autocrat's welfare-maximizing purge inference would select either the highest interior solution or the corner solution (as in our baseline). All our comparative statics would remain unchanged then (at the cost though of complicating the analysis). Selecting the lowest interior purge inference would

[^0]change some of our comparative statics, but imposing parameter values such that purge inference is continuous (as we do later) would reestablish them. As such, our results are robust to change in the equilibrium criterion.

## Proof of Corollary 1

Suppose $v(S, n c) \geq 0$. By Lemma 3, the purge is semi-indiscriminate if and only if (i) $r>$ $\mu^{S}\left(\kappa_{S}^{*}(L), L\right)=\lambda \frac{v(S, \tau)+V_{2}(c)+L}{\bar{v}+\overline{\bar{V}_{2}}+L}$ and (ii) $C_{1}, C_{0}$ satisfy $C_{1}<\frac{\beta\left(r-\lambda \frac{v(S, c)+V_{2}(c)+L}{\bar{v}+\overline{V_{2}}+L}\right) \mathcal{D}_{2}^{c, n c}-C_{0}}{1-\bar{v}-\overline{V_{2}}-L}$. Notice that as $v(S, c) \rightarrow v(S, n c), \mu^{S}\left(\kappa_{S}^{*}(L), L\right) \rightarrow \lambda$. Therefore, whenever $r>\lambda$, there exist $v(S, c)-v(S, n c)$, $C_{1}$, and $C_{0}$ sufficiently small so that both conditions are satisfied.

## B Proofs for Section 4

We first study the effect of an exogenous change in violence on beliefs.
Lemma B.1. Fixing the purge breadth,
(i) in a discriminate purge, $\mu^{F}\left(\kappa_{F}, L\right)$ is strictly decreasing in $L$;
(ii) in a semi-indiscriminate purge, $\mu^{S}\left(\kappa_{S}, L\right)$ is strictly decreasing in $L$.

Proof. Point (i). A similar reasoning as in Lemma 2 yields:

$$
\begin{equation*}
\frac{\partial \mu^{F}\left(\kappa_{F}, L\right)}{\partial L}=\frac{\lambda(1-\lambda)\left(1-e^{i}\left(\kappa_{F}, L ; c\right)\right)\left(1-e^{i}\left(\kappa_{F}, L ; n c\right)\right)}{\left(1-\bar{e}\left(\kappa_{F}, L\right)\right)^{2}}\left[\frac{\frac{\partial e^{i}\left(\kappa_{F}, L ; n c\right)}{\partial L}}{1-e^{i}\left(\kappa_{F}, L ; n c\right)}-\frac{\frac{\partial e^{i}\left(\kappa_{F}, L ; c\right)}{\partial L}}{1-e^{i}\left(\kappa_{F}, L ; c\right)}\right] \tag{B.1}
\end{equation*}
$$

Using agents' efforts Equation 5, $\frac{\partial e^{i}\left(\kappa_{F}, L ; c\right)}{\partial L}=\frac{\partial e^{i}\left(\kappa_{F}, L ; n c\right)}{\partial L}$. Since $e^{i}\left(\kappa_{F}, L ; n c\right)<e^{i}\left(\kappa_{F}, L ; c\right)$, $\frac{\partial \mu^{F}\left(\kappa_{F}, L\right)}{\partial L}<0$.
Point (ii). Regarding the autocrat's posterior after success, a similar reasoning as in Lemma 2 yields:

$$
\begin{equation*}
\frac{\partial \mu^{S}\left(\kappa_{S}, L\right)}{\partial L}=\frac{\lambda(1-\lambda) e^{i}\left(\kappa_{S}, L ; c\right) e^{i}\left(\kappa_{S}, L ; n c\right)}{\bar{e}\left(\kappa_{S}, L\right)^{2}}\left[\frac{\frac{\partial e^{i}\left(\kappa_{S}, L ; c\right)}{\partial L}}{e^{i}\left(\kappa_{S}, L ; c\right)}-\frac{\frac{\partial e^{i}\left(\kappa_{S}, L ; n c\right)}{\partial L}}{e^{i}\left(\kappa_{S}, L ; n c\right)}\right] \tag{B.2}
\end{equation*}
$$

Using agents' efforts Equation 5, $\frac{\partial e^{i}\left(\kappa_{S}, L ; c\right)}{\partial L}=\frac{\partial e^{i}\left(\kappa_{S}, L ; n c\right)}{\partial L}$. Since $e^{i}\left(\kappa_{S}, L ; n c\right)<e^{i}\left(\kappa_{S}, L ; c\right)$, $\frac{\partial \mu^{F}\left(\kappa_{F}, L\right)}{\partial L}<0$.

To facilitate the exposition, we use subscript $x$ to denote the partial derivative of some variable $z$ with respect to $x$ (i.e., $\partial z / \partial x=z_{x}$ ) and a similar notation for the second partial derivative. We also ignore superscript and arguments whenever possible

## Proof of Proposition 1

First, consider a partially discriminate purge. Observe that $K_{P D}\left(\kappa_{F}, L\right)=\beta\left(r-\mu^{F}\left(\kappa_{F}, L\right)\right) \mathcal{D}_{2}^{c, n c}-$ $\left(C_{0}+C_{1} \kappa_{F}\left(1-\left(\bar{v}+\kappa_{F}\left(\overline{V_{2}}+L\right)\right)\right)\right)$ is strictly increasing in $L$ using Lemma B.1. Since $\kappa_{F}^{*}(L)$ is defined as the solution to $K_{P D}\left(\kappa_{F}^{*}(L), L\right)=0$ by the Implicit Function Theorem we must have that $\kappa_{F}^{*}(L)$ is continuously and strictly increasing with $L$ (recall that $\partial K_{P D}\left(\kappa_{F}^{*}(L), L\right) / \partial \kappa_{F}<0$ from the proof of Lemma 3).

We now show that if Equation 8 holds, there exists $L^{\text {full }}<\bar{L}$ such that $\kappa_{F}^{*}(L)=1$ for all $L \geq L^{\text {full }}$. To see this, observe that for $L=\bar{L}$ and $\kappa_{F}=1$, we have $e^{i}(1, \bar{L}, c)=1$ so $\mu^{F}(1, \bar{L})=0$. Further, $K_{P D}(1, \bar{L})=\beta r \mathcal{D}^{c, n c}-\left(C_{0}+C_{1}\left(1-\left(\bar{v}+\overline{V_{2}}+\bar{L}\right)\right)\right)>0$ by Equation 8 (since $\left.\hat{\alpha}_{F}(\bar{L})=1-\bar{v}-\overline{V_{2}}-\bar{L}\right)$. By Lemma 3 (points (ii) and (iii)), the equilibrium purge breadth satisfies $\kappa_{F}^{*}(\bar{L})=1$. Since $\kappa_{F}^{*}(L)$ is strictly increasing with $L$ when interior, here exists $L^{\text {full }}<\bar{L}$ such that $\kappa_{F}^{*}(L)=1$ for all $L \geq L^{\text {full }}$ and $\kappa_{F}^{*}(L)<1$ otherwise. In turn, when Equation 8 does not hold, we must have $\kappa_{F}^{*}(\bar{L})<1$ which implies $\kappa_{F}^{*}(L)<1$ for all $L$. This completes the proof of part 1. and part 2.(i).
Suppose Equation 8 holds in what follows. For $L>L^{\text {full }}$, the purge is fully discriminate or semiindiscriminate. Recall that $K_{S D}\left(\kappa_{S}, L\right)=\beta\left(r-\mu^{S}\left(\kappa_{S}, L\right)\right) \mathcal{D}_{2}^{c, n c}-\left(C_{0}+C_{1}\left(1-\left(1-\kappa_{S}\right)^{2}\left(\bar{v}+\overline{V_{2}}+\right.\right.\right.$ $L))$ ) ). Observe that at $L=L^{\text {full }}, K_{S D}\left(0, L^{\text {full }}\right)<0\left(\right.$ since $\mu^{F}(\cdot)<\mu^{S}(\cdot)$ and $K_{P D}\left(1, L^{\text {full }}\right)=0>$ $K_{S D}\left(0, L^{\text {full }}\right)$. We need to consider two cases. Case (a) $K_{S D}(0, \bar{L}) \leq 0$ (e.g., this is always the case when $r \leq \lambda$ ). In this case, it is never profitable for the autocrat to purge from the success pool. Impose $L^{\text {ind }}:=\bar{L}$ then. Case (b) $K_{S D}(0, \bar{L})>0$. By Lemma B. 1 point (iii), it must then be that $\kappa_{S}^{*}(\bar{L})>0$. By Lemma B.1, $\mu^{S}\left(\kappa_{S}, L\right)$ is strictly increasing with $L$ so by the Implicit Function Theorem (recall that $\partial K_{S D}\left(\kappa_{S}^{*}(L), L\right) / \partial \kappa_{S}<0$ from the proof of Lemma 3), any interior equilibrium incidence $\kappa_{S}^{*}(L)$ is continuously and strictly increasing in $L$. In case (b), we thus obtain that there exists a unique $L^{\text {ind }} \in\left(L^{\text {full }}, \bar{L}\right)$ such that $\kappa_{S}^{*}(L)>0$ for all $L>L^{\text {ind }}$ and $\kappa_{S}^{*}(L)=0$ otherwise. This completes the proof of the proposition.

Recall that in what follows we impose: $C_{0}<\beta\left(r-\lambda \frac{1-\left(v(S, c)+V_{2}(c)\right)}{1-\left(\bar{v}+\overline{V_{2}}\right)}\right) \mathcal{D}^{c, n c}$ and $C_{0}+C_{1}\left(1-\left(\bar{v}+\overline{V_{2}}+\right.\right.$ $\bar{L}))<\beta r \mathcal{D}^{c, n c}\left(\right.$ i.e., Equation 8) so that $\kappa_{F}^{*}(0)>0$ and $L^{\text {full }} \in[0, \bar{L})$.

Lemma B.2. In a semi-indiscriminate purge $\left(\kappa_{F}^{*}(L) \in(0,1)\right)$, the purge incidence is convex in $L$.
Proof. Recall that $\kappa_{F}^{*}(L)$ is the solution to $K_{P D}\left(\kappa_{F}, L\right)=0$ with $K_{P D}\left(\kappa_{F}, L\right)$ defined in Equation A.5. Simple algebra yields (using $\mu_{\kappa_{F} \kappa_{F}}^{F}<0, \mu_{L L}^{F}<0$ and $\mu_{\kappa_{F} L}^{F}<0$ from $\mu^{F}=\frac{1-\left(v(S, c)+\kappa_{F}\left(V_{2}(c)+L\right)\right)}{1-\left(\bar{v}+\kappa_{F}\left(\overline{V_{2}}+L\right)\right)}$, see also the proof of Lemma C. 1 below): $\partial^{2} K_{P D}\left(\kappa_{F}, L\right) / \partial \kappa_{F}^{2}>0, \partial^{2} K_{P D}\left(\kappa_{F}, L\right) / \partial L^{2}>0$, and
$\partial^{2} K_{P D}\left(\kappa_{F}, L\right) / \partial \kappa_{F} \partial L>0$. Totally differentiating at $\kappa_{F}=\kappa_{F}^{*}(L)$, we obtain (ignoring arguments):

$$
\frac{\partial^{2} \kappa_{F}}{\partial L^{2}} \frac{\partial K_{P D}}{\partial \kappa_{F}}+\frac{\partial \kappa_{F}}{\partial L}\left(\frac{\partial^{2} K_{P D}}{\partial \kappa_{F}^{2}}+2 \frac{\partial^{2} K_{P D}}{\partial \kappa_{F} \partial L}\right)+\frac{\partial^{2} K_{P D}}{\partial L^{2}}=0
$$

Since $\partial K_{P D}\left(\kappa_{F}^{*}(L), L\right) / \partial \kappa_{F}<0$ (Lemma 3), $\frac{\partial^{2} \kappa_{F}}{\partial L^{2}}>0$.

## Proof of Proposition 2

In a partially discriminate purge $\left(\kappa_{F}^{*}(L) \in(0,1)\right)$, the total derivative of average effort with respect to violence is (using Equation 5):

$$
\begin{equation*}
\frac{d \bar{e}\left(\kappa_{F}^{*}(L), L\right)}{d L}=\frac{\partial \kappa_{F}^{*}(L)}{\partial L}\left(\overline{V_{2}}+L\right)+\kappa_{F}^{*}(L) \tag{B.3}
\end{equation*}
$$

From the proof of Proposition 1. $\frac{\partial \kappa_{F}^{*}(L)}{\partial L}>0$ so $\frac{d \bar{e}\left(\kappa_{F}^{*}(L), L\right)}{d L}>0$. Further given the convexity of $\kappa_{F}^{*}(L), \frac{d^{2} \bar{e}\left(\kappa_{F}^{*}(L), L\right)}{d L^{2}}>0$.
Suppose that there exists a unique solution to $K_{P D}\left(\kappa_{F}, L\right)=0$ (we provide a precise condition for this assumption to hold below, see Equation C.13. As $L \rightarrow L^{\text {full }}, \kappa_{F}^{*}(L) \rightarrow 1$. Since $\frac{\partial \kappa_{F}^{*}(L)}{\partial L}$ is continuous and increasing in $L$, so is $\frac{d e\left(\kappa_{F}^{*}(L), L\right)}{d L}$ and there exists a unique $L^{\text {fear }} \in\left[0, L^{\text {full }}\right)$ such that for all $L>L^{f e a r}, \frac{d \bar{e}\left(\kappa_{F}^{*}(L), L\right)}{d L}>1$. When there are multiple solutions, using Lemma 3, $\lim _{L \uparrow L^{\text {full }}} \kappa_{F}^{*}(L)<1$. If $\lim _{L \uparrow L^{\text {full }}} \frac{d \bar{e}\left(\kappa_{F}^{*}(L), L\right)}{d L}>1$, then there exists a unique $L^{\text {fear }}<L^{\text {full }}$ such that the claim holds (by convexity of average effort). If not, denote $L^{f e a r}=L^{\text {full }}$ which is uniquely defined. In a fully discriminate purge $\left(\kappa_{F}^{*}(L)=1, \kappa_{S}^{*}(L)=0\right)$, average effort is simply: $\bar{e}(1, L)=\bar{v}+\overline{V_{2}}+L$ so $\frac{d \bar{e}\left(\kappa_{F}^{*}(L), L\right)}{d L}=1$.

In a semi-indiscriminate purge $\left(\kappa_{S}^{*}(L) \in(0,1)\right)$, the total derivative of average effort with respect to violence is (using Equation 5):

$$
\begin{equation*}
\frac{d \bar{e}\left(\kappa_{S}^{*}(L), L\right)}{d L}=\frac{\partial\left(1-\kappa_{S}^{*}(L)\right)}{\partial L}\left(\bar{v}+\overline{V_{2}}+L\right)+\left(1-\kappa_{S}^{*}(L)\right) \tag{B.4}
\end{equation*}
$$

From the proof of Proposition 1 , $\frac{\partial \kappa_{S}^{*}(L)}{\partial L}>0$ so $\frac{d \bar{e}\left(\kappa_{S}^{*}(L), L\right)}{d L}<1$. It remains to show that $\frac{d \bar{e}\left(\kappa_{S}^{*}(L), L\right)}{d L}>0$. To see this, recall that $\kappa_{S}^{*}(L)$ is the solution to

$$
\begin{equation*}
C_{0}+C_{1}-C_{1}\left(1-\kappa_{S}\right)^{2}\left(\bar{v}+\overline{V_{2}}+L\right)=\beta\left(r-\lambda \frac{v(S, c)+V_{2}(c)+L}{\bar{v}+\overline{V_{2}}+L}\right) \mathcal{D}^{c, n c} \tag{B.5}
\end{equation*}
$$

We thus obtain:

$$
\frac{\partial\left(1-\kappa_{S}^{*}(L)\right)}{\partial L}=-\frac{\beta \lambda(1-\lambda)}{2 C_{1}\left(1-\kappa_{S}^{*}(L)\right)} \frac{(v(S, c)-v(S, n c))+\left(V_{2}(c)-V_{2}(n c)\right)}{\left(\bar{v}+\overline{V_{2}}+L\right)^{3}} \mathcal{D}^{c, n c}-\frac{\left(1-\kappa_{S}\right)}{2\left(\bar{v}+\overline{V_{2}}+L\right)}
$$

Plugging this into Equation B.4, we obtain

$$
\frac{d \bar{e}\left(\kappa_{S}^{*}(L), L\right)}{d L}=-\frac{\beta \lambda(1-\lambda)}{2 C_{1}\left(1-\kappa_{S}^{*}(L)\right)} \frac{(v(S, c)-v(S, n c))+\left(V_{2}(c)-V_{2}(n c)\right)}{\left(\bar{v}+\overline{V_{2}}+L\right)^{2}} \mathcal{D}^{c, n c}+\frac{1-\kappa_{S}^{*}(L)}{2}
$$

So after rearranging,
$\frac{d \bar{e}\left(\kappa_{S}^{*}(L), L\right)}{d L} \propto C_{1}\left(1-\kappa_{S}^{*}(L)\right)^{2}\left(\bar{v}+\overline{V_{2}}+L\right)-\beta \lambda(1-\lambda) \frac{(v(S, c)-v(S, n c))+\left(V_{2}(c)-V_{2}(n c)\right)}{\left(\bar{v}+\overline{V_{2}}+L\right)} \mathcal{D}^{c, n c}$
Denote $\mathcal{H}:=C_{1}\left(1-\kappa_{S}^{*}(L)\right)^{2}\left(\bar{v}+\overline{V_{2}}+L\right)-\beta \lambda(1-\lambda) \frac{(v(S, c)-v(S, n c))+\left(V_{2}(c)-V_{2}(n c)\right)}{\left(\bar{v}+\overline{V_{2}}+L\right)} \mathcal{D}^{c, n c}$. Using Equation B. 5 and our assumption that $C_{0}+C_{1}>\beta r \mathcal{D}^{c, n c}$, we obtain:

$$
\begin{aligned}
\mathcal{H} & =C_{0}+C_{1}-\beta\left(r-\lambda \frac{v(S, c)+V_{2}(c)+L}{\bar{v}+\overline{V_{2}}+L}\right) \mathcal{D}^{c, n c}-\beta \lambda(1-\lambda) \frac{(v(S, c)-v(S, n c))+\left(V_{2}(c)-V_{2}(n c)\right)}{\left(\bar{v}+\overline{V_{2}}+L\right)} \mathcal{D}^{c, n c} \\
& >\beta r \mathcal{D}^{c, n c}-\beta\left(r-\lambda \frac{v(S, c)+V_{2}(c)+L}{\bar{v}+\overline{V_{2}}+L}\right) \mathcal{D}^{c, n c}-\beta \lambda(1-\lambda) \frac{(v(S, c)-v(S, n c))+\left(V_{2}(c)-V_{2}(n c)\right)}{\left(\bar{v}+\overline{V_{2}}+L\right)} \mathcal{D}^{c, n c} \\
& =\frac{\beta \lambda}{\bar{v}+\overline{V_{2}}+L} \mathcal{D}^{c, n c}\left(v(S, c)+V_{2}(c)+L-(1-\lambda)\left((v(S, c)-v(S, n c))+\left(V_{2}(c)-V_{2}(n c)\right)\right)\right) \\
& =\beta \lambda \mathcal{D}^{c, n c}>0
\end{aligned}
$$

This completes the proof of the proposition.

## Proof of Proposition 3

In a partially discriminate purge $\left(\kappa_{F}^{*}(L) \in(0,1)\right)$, the purge breadth is $\kappa^{*}(L)=\alpha_{F}^{*}(L) \kappa_{F}^{*}(L)$ characterized by

$$
C_{0}+C_{1} \kappa^{*}(L)=\beta\left(r-\mu^{F}\left(\kappa_{F}^{*}(L), L\right)\right) \mathcal{D}^{c, n c}
$$

The total derivative of the posterior with respect to $L$ is $\frac{d \mu^{F}\left(\kappa_{F}^{*}(L), L\right)}{d L}=\frac{\partial \kappa_{F}^{*}(L)}{\partial L} \mu_{\kappa_{F}}^{F}\left(\kappa_{F}^{*}(L), L\right)+$ $\mu_{L}^{F}\left(\kappa_{F}^{*}(L), L\right)$, with $\frac{\partial \kappa_{F}^{*}(L)}{\partial L}>0$ (proof of Proposition 11, $\mu_{\kappa_{F}}^{F}<0$ (Lemma A.1, and $\mu_{L}^{F}<0$ (Lemma B.1). So $\frac{d \mu^{F}\left(\kappa_{F}^{*}(L), L\right)}{d L}<0$ and by the Implicit Function Theorem, $d \kappa^{*}(L) / d L>0$.
In a fully discriminate purge $\left(\kappa_{F}^{*}(L)=1\right.$ and $\left.\kappa_{S}^{*}(L)=0\right), \kappa^{*}(L)=1-\bar{e}(1, L)=1-\left(\bar{v}+\overline{V_{2}}+L\right)$ strictly decreasing with $L$.
In a semi-discriminate purge, the purge breadth is $\kappa^{*}(L)=1 \times \alpha_{F}^{*}(L)+\kappa_{S}^{*}(L) \times \alpha_{S}^{*}(L)$ characterized by

$$
C_{0}+C_{1} \kappa^{*}(L)=\beta\left(r-\mu^{S}\left(\kappa_{S}^{*}(L), L\right)\right) \mathcal{D}^{c, n c}
$$

The total derivative of the posterior with respect to $L$ is $\frac{d \mu^{S}\left(\kappa_{S}^{*}(L), L\right)}{d L}=\frac{\partial \kappa_{S}^{*}(L)}{\partial L} \mu_{\kappa_{S}}^{S}\left(\kappa_{S}^{*}(L), L\right)+$ $\mu_{L}^{S}\left(\kappa_{S}^{*}(L), L\right)$, with $\frac{\partial \kappa_{S}^{*}(L)}{\partial L}>0$ (proof of Proposition 1), $\mu_{\kappa_{S}}^{S}=0$ (Lemma A.1), and $\mu_{L}^{S}<0$ (Lemma B.1. So $\frac{d \mu^{S}\left(\kappa_{S}^{*}(L), L\right)}{d L}<0$ and by the Implicit Function Theorem, $d \kappa^{*}(L) / d L>0$.

Combining the analysis above with Proposition 1 yields the claim.

To ease the exposition, we ignore the equilibrium '*' superscript and arguments whenever possible in what follows.

## Proof of Proposition 4

Point (i). For $L<L^{\text {full }}$, the proportion of ideologues in the pool of survivors is:

$$
\begin{align*}
\mathcal{S}(L) & =\frac{(1-\bar{e})\left(1-\kappa_{F}\right) \mu^{F}+\bar{e} \mu^{S}}{1-(1-\bar{e}) \kappa_{F}} \\
& =\frac{\lambda-(1-\bar{e}) \mu^{F} \kappa_{F}}{1-(1-\bar{e}) \kappa_{F}} \tag{B.6}
\end{align*}
$$

Consider the function $F(x)=\frac{1-x \mu^{F}}{1-x}$, its derivative is $F^{\prime}(x)=\frac{\left(\lambda-\mu^{F}\right)}{(1-x)^{2}}>0$.
We obtain

$$
\begin{equation*}
\frac{d \mathcal{S}(L)}{d L}=\lambda \frac{d\left(1-\bar{e}\left(\kappa_{F}(L), L\right)\right) \kappa_{F}(L)}{d L} F^{\prime}\left(\left(1-\bar{e}\left(\kappa_{F}(L), L\right) \kappa_{F}(L)\right)-\frac{d \mu^{F}\left(\kappa_{F}(L), L\right)}{d L} \frac{(1-\bar{e}) \kappa_{F}}{1-(1-\bar{e}) \kappa_{F}}\right. \tag{B.7}
\end{equation*}
$$

From the proof of Proposition 3, we know that $\left(1-\bar{e}\left(\kappa_{F}(L), L\right)\right) \kappa_{F}(L)=\kappa(L)$ and $d \kappa(L) / d L>0$, and $\frac{d \mu^{F}\left(\kappa_{F}(L), L\right)}{d L}<0$. Hence $\frac{d \mathcal{S}(L)}{d L}>0$
For $L \geq L^{\text {full }}$, the proportion of ideologues in the pool of survivors is

$$
\begin{equation*}
\mathcal{S}(L)=\mu^{S} \tag{B.8}
\end{equation*}
$$

Since $\mu_{L}^{S}<0$ (Lemma B.1), $\frac{d \mathcal{S}(L)}{d L}<0$.
Points (ii) and (iii). For $L<L^{\text {full }}$, the proportion of congruent subordinates in the second period is:

$$
\begin{align*}
\mathcal{P}(L) & =(1-\bar{e}) \kappa_{F} r+(1-\bar{e})\left(1-\kappa_{F}\right) \mu^{F}+\bar{e} \mu^{S} \\
& =(1-\bar{e}) \kappa_{F}\left(r-\mu^{F}\right)+\lambda \tag{B.9}
\end{align*}
$$

As above $d(1-\bar{e}) \kappa_{F} / d L>0$ and $d \mu^{F} / d L<0$ so $\frac{d \mathcal{P}(L)}{d L}>0$.
For $L \in\left[L^{\text {full }}, L^{\text {ind }}\right)$, the proportion of congruent subordinates in the second period is:

$$
\begin{equation*}
\mathcal{P}(L)=(1-\bar{e}) r+\bar{e} \mu^{S} \tag{B.10}
\end{equation*}
$$

Using $\mu^{S}=\lambda \frac{v(S, c)+V_{2}(c)+L}{\bar{v}+\overline{V_{2}}+L}$ and $\bar{e}=\bar{v}+\overline{V_{2}}+L$, we obtain: $\mathcal{P}(L)=\left(1-\left(\bar{v}+\overline{V_{2}}+L\right)\right) r+\lambda(v(S, c)+$ $\left.V_{2}(c)+L\right)$ and

$$
\begin{equation*}
\frac{d \mathcal{P}(L)}{d L}=\lambda-r \tag{B.11}
\end{equation*}
$$

$\mathcal{P}_{L}(L) \geq 0 \Leftrightarrow \lambda \geq r$. Further whenever $r \leq \lambda, L^{\text {ind }}=\bar{L}$ so we obtain point (ii).
To prove point (iii), suppose that $L^{\text {ind }}<\bar{L}$ (otherwise, the claim holds directly) and $L \geq L^{\text {ind }}$, the proportion of ideologues in the party in the second period is:

$$
\begin{align*}
\mathcal{P}(L) & =\left((1-\bar{e})+\bar{e} \kappa_{S}\right) r+\bar{e}\left(1-\kappa_{S}\right) \mu^{S} \\
& =r-\bar{e}\left(1-\kappa_{S}\right)\left(r-\mu^{S}\right) \tag{B.12}
\end{align*}
$$

Recall that $\kappa_{S}^{*}$ is the solution to $C_{0}+C_{1}\left(1-\bar{e}\left(1-\kappa_{S}\right)\right)=\beta\left(r-\mu^{S}\right) \mathcal{D}^{c, n c}$ so $\frac{d \bar{e}\left(1-\kappa_{S}\right)}{d L}=\frac{\beta}{C_{1}} \frac{d \mu^{S}}{d L} \mathcal{D}^{c, n c}$. Hence, we obtain:

$$
\begin{aligned}
\frac{d \mathcal{P}(L)}{d L} & =-\frac{d \bar{e}\left(1-\kappa_{S}\right)}{d L}\left(r-\mu^{S}\right)+\frac{d \mu^{S}}{d L} \bar{e}\left(1-\kappa_{S}\right) \\
& =-\frac{d \mu^{S}}{d L} \frac{\beta}{C_{1}}\left(r-\mu^{S}\right) \mathcal{D}^{c, n c}+\frac{d \mu^{S}}{d L} \bar{e}\left(1-\kappa_{S}\right) \\
& =\frac{d \mu^{S}}{d L} \frac{1}{C_{1}}\left(C_{1} \bar{e}\left(1-\kappa_{S}\right)-\beta\left(r-\mu^{S}\right) \mathcal{D}^{c, n c}\right) \\
& =\frac{d \mu^{S}}{d L}\left(C_{0}+C_{1}-2 \beta\left(r-\mu^{S}\right) \mathcal{D}^{c, n c}\right)
\end{aligned}
$$

Recall that by assumption $C_{0}+C_{1}>\beta r \mathcal{D}^{c, n c}$ so $C_{0}+C_{1}-2 \beta\left(r-\mu^{S}\right) \mathcal{D}^{c, n c}>\beta\left(2 \mu^{S}-r\right) \mathcal{D}^{c, n c}$. Since $\lambda<\mu^{S}, C_{0}+C_{1}-2 \beta\left(r-\mu^{S}\right) \mathcal{D}^{c, n c}>0$ for all $r \in\left(\lambda, 2 \lambda\right.$. Given $\frac{d \mu^{S}}{d L}<0$ (Lemma B.1), we obtain $\frac{d \mathcal{P}(L)}{d L}<0$ as claimed.

## C Proofs for Section 5

Denote $B(L)$ the autocrat's expected benefit from violence. The next Lemmas characterize the properties of $B(L)$ ignoring superscript and arguments whenever possible. In what follows, we focus on the case when $L^{\text {ind }}<\bar{L}$ (notice that this implies $r>\lambda$ ). The analysis can easily be extended to the case when $L^{\text {ind }}=\bar{L}$.

Lemma C.1. The expected benefit of violence is $\mathcal{C}^{\infty}$, strictly increasing, and convex in $L$ for $L \leq L^{\text {full }}$.

Proof. For $L \leq L^{\text {full }}$, the expected benefit of violence is:

$$
\begin{align*}
B(L)= & \bar{e}+\beta(1-\bar{e})\left(\kappa_{F}\left(r W_{2}(c)+(1-r) W_{2}(n c)\right)+\left(1-\kappa_{F}\right)\left(\mu^{F} W_{2}(c)+\left(1-\mu^{F}\right) W_{2}(n c)\right)\right) \\
& +\beta \bar{e}\left(\mu^{S} W_{2}(c)+\left(1-\mu^{S}\right) W_{2}(n c)\right)-C\left((1-\bar{e}) \kappa_{F}\right) \\
B(L)= & \bar{e}+\beta\left(\lambda W_{2}(c)+(1-\lambda) W_{2}(n c)\right)+\beta(1-\bar{e}) \kappa_{F}\left(r-\mu^{F}\right) \mathcal{D}^{c, n c}-C\left((1-\bar{e}) \kappa_{F}\right) \tag{C.1}
\end{align*}
$$

Since in the interval $\left[0, L^{\text {full }}\right]$, all functions are $\mathcal{C}^{\infty}$ so is $B(L)$.
Making use of the Envelop Theorem as $\kappa_{F}^{*}$ is interior, we obtain:

$$
\begin{equation*}
\frac{d B(L)}{d L}=\frac{d \bar{e}\left(\kappa_{F}^{*}(L), L\right)}{d L}-\beta(1-\bar{e}) \kappa_{F} \frac{d \mu^{F}\left(\kappa_{F}^{*}(L), L\right)}{d L} \mathcal{D}^{c, n c} \tag{C.2}
\end{equation*}
$$

Since $\frac{d \bar{e}\left(\kappa_{F}^{*}(L), L\right)}{d L}>0$ (Proposition 2) and $\frac{d \mu^{F}\left(\kappa_{F}^{*}(L), L\right)}{d L}<0$ (proof of Proposition 3), $d B(L) / d L>0$. To prove that the expected benefit of violence is strictly convex, we proceed in three steps. First, we compute the second (total) derivative of the marginal benefit of violence. Second, we look at the second (partial) derivatives of effort and autocrat's posterior with respect to $L$ and $\kappa_{F}$. The last step proves the claim.
Step 1. Using Equation C.2, we obtain:

$$
\begin{equation*}
\frac{d^{2} B(L)}{d L^{2}}=\frac{d^{2} \bar{e}\left(\kappa_{F}^{*}(L), L\right)}{d L^{2}}-\beta(1-\bar{e}) \kappa_{F} \frac{d^{2} \mu^{F}\left(\kappa_{F}^{*}(L), L\right)}{d L^{2}} \mathcal{D}^{c, n c}-\beta \frac{d(1-\bar{e}) \kappa_{F}}{d L} \frac{d \mu^{F}\left(\kappa_{F}^{*}(L), L\right)}{d L} \mathcal{D}^{c, n c} \tag{C.3}
\end{equation*}
$$

with

$$
\begin{align*}
\frac{d^{2} \bar{e}\left(\kappa_{F}^{*}(L), L\right)}{d L^{2}} & =2 \frac{\partial \kappa_{F}^{*}(L)}{\partial L}+\frac{\partial^{2} \kappa_{F}^{*}(L)}{\partial L^{2}}\left(\overline{V_{2}}+L\right)  \tag{C.4}\\
\frac{d^{2} \mu^{F}\left(\kappa_{F}^{*}(L), L\right)}{d L^{2}} & =\mu_{L L}^{F}+2 \frac{\partial \kappa_{F}^{*}(L)}{\partial L} \mu_{\kappa_{F} L}^{F}+\left(\frac{\partial \kappa_{F}^{*}(L)}{\partial L}\right)^{2} \mu_{\kappa_{F} \kappa_{F}}^{F}+\frac{\partial^{2} \kappa_{F}^{*}(L)}{\partial L^{2}} \mu_{\kappa_{F}}^{F} \tag{C.5}
\end{align*}
$$

Step 2. Using $\mu^{F}=\lambda \frac{1-e(i)}{1-\bar{e}}$, we obtain for $j \in\{\kappa, L\}$ :

$$
\begin{align*}
\mu_{j j}^{F}= & \frac{\lambda(1-\lambda)(1-e(c))(1-e(n c))}{(1-\bar{e})^{3}} \times\left[\left(\frac{e_{j j}(n c)}{(1-e(n c))}-\frac{e_{j j}(c)}{(1-e(c))}\right)(1-\bar{e})+2 \bar{e}_{j}\left(\frac{e_{j}(n c)}{1-e(n c)}-\frac{e_{j}(c)}{1-e(c)}\right)\right] \\
\mu_{\kappa_{F} L}^{F}= & \frac{\lambda(1-\lambda)(1-e(c))(1-e(n c))}{(1-\bar{e})^{3}} \times\left[\left(\frac{e_{\kappa_{F} L}(n c)}{(1-e(n c))}-\frac{e_{\kappa_{F} L}(c)}{(1-e(c))}\right)(1-\bar{e})+\bar{e}_{L}\left(\frac{e_{\kappa_{F}}(n c)}{1-e(n c)}-\frac{e_{\kappa_{F}}(c)}{1-e(c)}\right)\right] \\
& +\frac{\lambda(1-\lambda)}{(1-\bar{e})^{3}}\left[\bar{e}_{L}\left(e_{\kappa_{F}}(n c)(1-e(c))-e_{\kappa_{F}}(c)(1-e(n c))\right)+(1-\bar{e})\left(e_{\kappa_{F}}(c) e_{L}(n c)-e_{L}(c) e_{\kappa_{F}}(n c)\right)\right] \\
= & \frac{\lambda(1-\lambda)(1-e(c))(1-e(n c))}{(1-\bar{e})^{3}} \times\left[\left(\frac{e_{\kappa_{F} L}(n c)}{(1-e(n c))}-\frac{e_{\kappa_{F} L}(c)}{(1-e(c))}\right)(1-\bar{e})+\bar{e}_{L}\left(\frac{e_{\kappa_{F}}(n c)}{1-e(n c)}-\frac{e_{\kappa_{F}}(c)}{1-e(c)}\right)\right] \\
& +\frac{\lambda(1-\lambda)}{(1-\bar{e})^{3}}\left[e_{\kappa_{F}}(n c)\left(\bar{e}_{L}(1-e(c))-e_{L}(c)(1-\bar{e})\right)+e_{\kappa_{F}}(c)\left(e_{L}(n c)(1-\bar{e})-\bar{e}_{L}(1-e(n c))\right)\right] \\
\mu_{\kappa_{F} L}^{F}= & \frac{\lambda(1-\lambda)(1-e(c))(1-e(n c))}{(1-\bar{e})^{3}} \times\left[\left(\frac{e_{\kappa_{F} L}(n c)}{(1-e(n c))}-\frac{e_{\kappa_{F} L}(c)}{(1-e(c))}\right)(1-\bar{e})+\bar{e}_{L}\left(\frac{e_{\kappa_{F}}(n c)}{1-e(n c)}-\frac{e_{\kappa_{F}}(c)}{1-e(c)}\right)\right] \\
& +\frac{\lambda(1-\lambda)(1-e(c))(1-e(n c))}{(1-\bar{e})^{3}}\left[\left((1-\lambda) e_{\kappa_{F}}(n c)+\lambda e_{\kappa_{F}}(c)\right)\left(\frac{e_{L}(n c)}{(1-e(n c))}-\frac{e_{L}(c)}{(1-e(c))}\right)\right] \tag{C.7}
\end{align*}
$$

Using $e(\tau)=v(\tau)+\kappa_{F}\left(V_{2}(\tau)+L\right)$, we obtain:

$$
\begin{array}{r}
e_{\kappa_{F} \kappa_{F}}(\tau)=0 \\
e_{L L}(\tau)=0 \\
e_{\kappa_{F} L}(\tau)=1
\end{array}
$$

This implies that $\mu_{L L}^{F}<0\left(\mu_{\kappa_{F} \kappa_{F}}^{F}<0\right.$ by Lemma A.1) and $\mu_{L \kappa_{F}}^{F}<0($ since $e(c)>e(n c)$ and $\left.e_{j}(c) \geq e_{j}(n c), j \in\left\{\kappa_{F}, L\right\}\right)$.

Step 3. Plugging all partial derivatives into Equation C. 4 and Equation C. 5 and given that $\kappa_{F}^{*}(L)$ is convex in $L$ (Lemma B.2, we obtain: $\frac{d^{2} \bar{e}\left(\kappa_{F}^{*}(L), L\right)}{d L^{2}}>0$ and $\frac{d^{2} \mu^{F}\left(\kappa_{F}^{*}(L), L\right)}{d L^{2}}<0$. Given that $\frac{d(1-\bar{e}) \kappa_{F}}{d L}>0$ and $\frac{d(1-\bar{e}) \kappa_{F}}{d L} \frac{d \mu^{F}\left(\kappa_{F}^{*}(L), L\right)}{d L}<0$ (see Proposition 3), Equation C.3 yields $\frac{d^{2} B(L)}{d L^{2}}>0$.

Lemma C.2. The expected benefit of violence is $\mathcal{C}^{\infty}$, strictly increasing, and strictly concave in $L$ for $L \in\left(L^{\text {full }}, L^{\text {ind }}\right]$.

Proof. For $L \in\left(L^{\text {full }}, L^{\text {ind }}\right]$, the expected benefit of violence is:

$$
\begin{equation*}
B(L)=\bar{e}+\beta(1-\bar{e})\left(r W_{2}(c)+(1-r) W_{2}(n c)\right)+\beta \bar{e}\left(\mu^{S} W_{2}(c)+\left(1-\mu^{S}\right) W_{2}(n c)\right)-C(1-\bar{e}) \tag{C.8}
\end{equation*}
$$

with $e(\tau)=v(\tau)+V(\tau)+L, \tau \in\{c, n c\}$. Taking the derivative, we obtain:

$$
\begin{equation*}
\frac{d B(L)}{d L}=1+\beta(\lambda-r) \mathcal{D}^{c, n c}+C^{\prime}(1-\bar{e}) \tag{C.9}
\end{equation*}
$$

Since $v(S, \tau)<1, \tau \in\{c, n c\}$ and $\mathcal{D}^{c, n c}=v(S, c)-\max \{0, v(S, n c)\}<1$, we obtain: $\frac{d B(L)}{d L}>0$. Further, from Equation C.9,

$$
\begin{equation*}
\frac{d^{2} B(L)}{d L^{2}}=-C^{\prime \prime}(1-\bar{e})<0 \tag{C.10}
\end{equation*}
$$

For completeness, when $L \geq L^{\text {ind }}$, the expected benefit of violence is:

$$
\begin{align*}
B(L)= & \bar{e}+\beta(1-\bar{e})\left(r W_{2}(c)+(1-r) W_{2}(n c)\right) \\
& +\beta \bar{e}\left(\kappa_{S}\left(r W_{2}(c)+(1-r) W_{2}(n c)\right)+\left(1-\kappa_{S}\right)\left(\mu^{S} W_{2}(c)+\left(1-\mu^{S}\right) W_{2}(n c)\right)\right)-C\left((1-\bar{e})+\kappa_{S} \bar{e}\right) \\
B(L)= & \left.\bar{e}+\beta \mu^{S} \mathcal{D}^{c, n c}+\beta\left((1-\bar{e})+\kappa_{S} \bar{e}\right)\right)\left(r-\mu^{S}\right) \mathcal{D}^{c, n c}-C\left((1-\bar{e})+\bar{e} \kappa_{S}\right) \tag{C.11}
\end{align*}
$$

Taking the derivative and using the Envelop Theorem, we obtain:

$$
\begin{equation*}
\frac{d B(L)}{d L}=\frac{d \bar{e}}{d L}+\beta \frac{d \mu^{S}}{d L} \bar{e}\left(1-\kappa_{S}\right) \mathcal{D}^{c, n c} \tag{C.12}
\end{equation*}
$$

From Proposition 1, $L^{\text {full }}$ is the unique solution to $K_{P D}(1, L)=0 \Leftrightarrow C_{0}+C_{1}\left(1-\left(\bar{v}+\overline{V_{2}}+L\right)\right)=$ $\beta\left(r-\mu^{F}(1, L)\right) \mathcal{D}^{c, n c}$. The next Lemma establishes a necessary and sufficient condition such that $B(L)$ is continuous in $L$.

Lemma C.3. The expected benefit of violence $B(L)$ is continuous if and only if

$$
\begin{equation*}
\frac{\partial K_{P D}\left(1, L^{\text {full }}\right)}{\partial \kappa_{F}} \leq 0 \tag{C.13}
\end{equation*}
$$

Proof. If Equation C. 13 does not hold, it must be that at $L=L^{\text {full }}$, there exists two solutions to $K_{P D}\left(\kappa_{F}, L^{\text {full }}\right)=0: \kappa_{F}^{\prime}(L) \in(0,1)$ and $\kappa_{F}^{\prime \prime}(L)=1$. Using the proof of Lemma 3, we then have that $\lim _{L \uparrow L^{\text {full }}} \kappa_{F}^{*}(L)<1=\lim _{L \downarrow L^{\text {full }}} \kappa_{F}^{*}(L)$. Rearranging Equation C.1, we obtain for $L<L^{\text {full }}$ :

$$
B(L)=\bar{e}+\beta W_{2}(n c)+\beta \lambda \mathcal{D}^{c, n c}+\beta(1-\bar{e}) \kappa_{F}\left(r-\mu^{F}\right) \mathcal{D}^{c, n c}-C\left((1-\bar{e}) \kappa_{F}\right)
$$

Rearranging Equation C.8, we obtain for $L \geq L^{\text {full }}$ (using $\mu^{F}=\lambda \frac{1-e(c)}{1-\bar{e}}$ and $\mu^{S}=\lambda \frac{e(c)}{\bar{e}}$ ):

$$
\begin{aligned}
B(L) & =\bar{e}+\beta(1-\bar{e}) W_{2}(n c)+\beta(1-\bar{e}) r \mathcal{D}^{c, n c}+\beta \bar{e} W_{2}(n c)+\beta \bar{e} \mu^{S} \mathcal{D}^{c, n c}-C(1-\bar{e}) \\
& =\bar{e}+\beta W_{2}(n c)+\beta(1-\bar{e}) r \mathcal{D}^{c, n c}+\beta \lambda e(c) \mathcal{D}^{c, n c}-C(1-\bar{e}) \\
& =\bar{e}+\beta W_{2}(n c)+\beta(1-\bar{e}) r \mathcal{D}^{c, n c}+\beta \lambda \mathcal{D}^{c, n c}-\beta \lambda(1-e(c)) \mathcal{D}^{c, n c}-C(1-\bar{e}) \\
& =\bar{e}+\beta W_{2}(n c)+\beta \lambda \mathcal{D}^{c, n c}+\beta(1-\bar{e})\left(r-\mu^{F}\right) \mathcal{D}^{c, n c}-C((1-\bar{e}))
\end{aligned}
$$

For any interior solution for the purge inference $k_{F}^{*}(L)$, using the quadratic cost of purging, we obtain $B(L)=\bar{e}+\beta W_{2}(n c)+\beta \lambda \mathcal{D}^{c, n c}+\frac{C_{1}}{2}\left((1-\bar{e}) \kappa_{F}^{*}(L)\right)^{2}$. Since at $L=L^{\text {full }}$, both $\kappa_{F}^{\prime}(L) \in(0,1)$ and $\kappa_{F}^{\prime \prime}(L)=1$ are solution of $K_{P D}\left(\kappa_{F}, L\right)=0$ and $\bar{e}\left(\kappa_{F}^{\prime}\left(L^{\text {full }}\right), L^{\text {full }}\right)<\bar{e}\left(\kappa_{F}^{\prime \prime}\left(L^{\text {full }}\right), L^{\text {full }}\right)$, we obtain (slightly abusing notation) that $\left.B(L)\right|_{\kappa_{F}=\kappa_{F}^{\prime}(L)}<\left.B(L)\right|_{\kappa_{F}=\kappa_{F}^{\prime \prime}(L)}$. Consequently, $\lim _{L \uparrow L^{\text {full }}} B(L)<$ $\lim _{L \downarrow L^{\text {full }}} B(L)$ and $B(L)$ is not continuous.
In turn, suppose Equation C. 13 holds. Then $\kappa_{F}^{*}\left(L^{\text {full }}\right)=1$ is the unique solution to $K_{P D}\left(\kappa_{F}, L^{\text {full }}\right)=$ 0 and $\kappa_{F}^{*}(L)$ is continuous in $L$ for all $L$. As a result, after rearranging Equation C. 1 and Equation C. 8 as well as using $\bar{e}\left(1, L^{\text {full }}\right) \mu^{S}\left(1, L^{\text {full }}\right)=\lambda e\left(1, L^{\text {full }} ; i\right)$ and $\left(1-\bar{e}\left(1, L^{\text {full }}\right)\right) \mu^{F}\left(1, L^{\text {full }}\right)=$ $\lambda\left(1-e\left(1, L^{\text {full }} ; c\right)\right)$, we obtain $\lim _{L_{\uparrow} L^{f u l l}} B(L)=\bar{e}\left(1, L^{f u l l}\right)+\beta W_{2}(n c)+\beta\left(\left(1-\bar{e}\left(1, L^{f u l l}\right)\right) r+\lambda e\left(1, L^{f u l l} ; c\right)\right) \mathcal{D}^{c, n c}-$ $C\left(1-\bar{e}\left(1, L^{\text {full }}\right)\right)=\lim _{L \downarrow L^{\text {full }}} B(L)$. Comparing Equation C.8 and Equation C.11 and using $\lim _{L \rightarrow L^{\text {ind }}} \kappa_{S}^{*}(L)=$ 0, we obtain: $\lim _{L \uparrow L^{i n d}} B(L)=\bar{e}\left(1, L^{\text {ind }}\right)+\beta W_{2}(n c)+\beta\left(\left(1-\bar{e}\left(1, L^{\text {ind }}\right)\right) r-\lambda e\left(1, L^{\text {ind }} ; c\right)\right) \mathcal{D}^{c, n c}-C(1-$ $\left.\bar{e}\left(1, L^{\text {ind }}\right)\right)=\lim _{L \downarrow L^{\text {ind }}} B(L) . B(L)$ is then continuous for all $L$.

In what follows, we assume that Equation C. 13 holds. Observe that since the properties of $B(L)$ (Lemmas C. 1 and C.2) do not depend on the continuity of $B(L)$, the analysis below remains valid. To find the equilibrium intensity of violence when Equation C. 13 does not hold, in addition to the analysis below, it is necessary to consider cases when the marginal cost interacts the marginal benefit before and after the discontinuity at $L=L^{f u l l}$. Consequently, assuming that $B(L)$ is continuous simply limits the number of cases to be analyzed. ${ }^{2}$ We nonetheless establishes existence and (generically) uniqueness of an equilibrium when Equation C. 13 does not hold and $B(L)$ is not continuous in Remark C. 1 below.

Lemma C.4. The marginal benefit of violence satisfies:

$$
\lim _{L \uparrow L^{f u l l}} \frac{d B(L)}{d L}>\frac{d B\left(L_{1}\right)}{d L}>\frac{d B\left(L_{2}\right)}{d L} \text { for all } L_{1} \in\left(L^{\text {full }}, L^{\text {ind }}\right] \text { and } L_{2} \in\left(L^{i n d}, \bar{L}\right]
$$

Proof. The proof of a discontinuity in the marginal benefit at $L=L^{\text {full }}$ proceeds in three steps. First, we show that $-\kappa(L) \frac{d \mu^{F}}{d L}>\lambda-\mu^{F}$ as $L \uparrow L^{\text {full }}$. Second, we show that $\frac{d B(L)}{d L}=1+\beta(\lambda-$ $\left.\mu^{F}\right) W_{2}(i)$ as $L \downarrow L^{\text {full }}$, Finally, we prove this part of the claim.
Step 1. Using the definition of $\mu^{F},-(1-\bar{e}) \kappa_{F}(L) \frac{d \mu^{F}}{d L}>-\kappa_{F}(L)(1-\bar{e}) \mu_{L}^{F}$ for all $L \in\left[0, L^{\text {full }] \text {, we }}\right.$

[^1]obtain
\[

$$
\begin{aligned}
-(1-\bar{e}) \kappa_{F}(L) \frac{d \mu^{F}}{d L} & >\kappa_{F}(L)\left((1-\bar{e}) \lambda \frac{e_{L}(c)(1-\bar{e})-\bar{e}_{L}(1-e(c))}{(1-\bar{e})^{2}}\right) \\
& =\kappa_{F}(L)\left(\lambda e_{L}(c)-\mu^{F} \bar{e}_{L}\right)
\end{aligned}
$$
\]

As $e_{L}(\tau)=\kappa_{F}(L)$ Equation 5) and $\kappa_{F}(L) \xrightarrow{L \uparrow L^{\text {full }}} 1$, we obtain that $\kappa_{F}(L)\left(\lambda e_{L}(c)-\mu^{F} \bar{e}_{L}\right) \xrightarrow{L \uparrow L^{\text {full }}}$ $\lambda-\mu^{F}$.
Step 2. As $L \downarrow L^{\text {full }}, C^{\prime}(1-\bar{e})=\beta\left(r-\mu^{F}\right) \mathcal{D}^{c, n c}$. Hence, we can rewrite Equation C. 9 as $L \downarrow L^{\text {full }}$ as (slightly abusing notation by using equalities):

$$
\begin{aligned}
\frac{d B(L)}{d L} & =1+\beta(\lambda-r) \mathcal{D}^{c, n c}+\beta\left(r-\mu^{F}\right) \mathcal{D}^{c, n c} \\
& =1+\beta\left(\lambda-\mu^{F}\right) \mathcal{D}^{c, n c}
\end{aligned}
$$

Step 3. From Proposition 2 , $\lim _{L \uparrow L^{\text {full }}} \frac{d \bar{e}}{d L}>1$. Using Equation C.2 and step 2, as $L \uparrow L^{\text {full }}$

$$
\begin{aligned}
\frac{d B(L)}{d L}= & \frac{d \bar{e}}{d L}-\beta(1-\bar{e}) \frac{d \mu^{F}}{d L} \mathcal{D}^{c, n c} \\
& >1+\beta\left(r-\mu^{F}\right) \mathcal{D}^{c, n c}
\end{aligned}
$$

which proves $\lim _{L \uparrow L^{f u l l}} \frac{d B(L)}{d L}>\lim _{L \downarrow L^{\text {full }}} \frac{d B(L)}{d L}$.
Since $B(L)$ is strictly concave for $L \in\left(L^{\text {full }}, L^{\text {ind }}\right]$, this directly implies that $\lim _{L \uparrow L^{f u l l}} \frac{d B(L)}{d L}>\frac{d B\left(L_{1}\right)}{d L}$ for all $L_{1} \in\left(L^{\text {full }}, L^{\text {ind }}\right]$.
We now consider the discontinuity in the marginal benefit around $L^{\text {ind }}$. By Equation C.9, $\frac{d B\left(L_{1}\right)}{d L}>1$ for all $\in\left(L^{f u l l}, L^{\text {ind }}\right]$. By Equation C.12, $\frac{d B\left(L_{2}\right)}{d L}<1$ for all $L_{2}>L^{\text {ind }}$ since $d \bar{e}(L) / d L<1$ (Proposition 2) and the other term is negative.

## Proof of Proposition 5

Existence follows from the fact that $B(L)$ is continuous (Lemma C.3) and the maximization problem is over a compact set $[0, \bar{L}]$. We now look at different cases to characterize the maximum

Case 1. Suppose at $L=L^{\text {full }}, \zeta_{0}+\zeta_{1} L \geq 1+\beta \mathcal{D}^{c, n c}\left(\lambda-\mu^{F}\right)$ (point (i)). Recall that $\lim _{L \downarrow L^{f u l l}} B(L)=$ $1+\beta \mathcal{D}^{c, n c}\left(\lambda-\mu^{F}\right)$ (proof of Lemma C.4). By Lemma C.4, it must then be that $\zeta_{0}+\zeta_{1} L>B(L)$ for all $L>L^{\text {full }}$. Hence, $L^{*}<L^{\text {full }}$. Since $B(L)$ is convex over the interval $\left[0, L^{\text {full }] \text {, we need to }}\right.$ consider two cases at $L=L^{\text {full }}$ : (a) $\zeta_{0}+\zeta_{1} L \geq \frac{d B(L)}{d L}$ and (b) $\zeta_{0}+\zeta_{1} L<\frac{d B(L)}{d L}$. In case (a), the solution is unique and equals to either 0 or the unique solution to $\zeta_{0}+\zeta_{1} L=\frac{d B(L)}{d L}$. In case (b), $L=L^{\text {full }}$ is always a corner solution. If $\zeta_{0}+\zeta_{1} L=\frac{d B(L)}{d L}$ has no solution or a single solution (which
must be a local minimum) in the interval $\left[0, L^{f u l l}\right]$, then $L^{*}=L^{\text {full }}$. If $\zeta_{0}+\zeta_{1} L=\frac{d B(L)}{d L}$ admits two solutions, then the smallest solution $L^{1}$ is a local maximum, whereas the highest solution $L^{2}$ is a local minimum. The autocrat then compares $B\left(L^{1}\right)$ and $B\left(L^{f u l l}\right)$ and there generically exists a unique solution unless $B\left(L^{1}\right)=B\left(L^{\text {full }}\right)$.
Case 2. Suppose at $L=L^{\text {full }}, \zeta_{0}+\zeta_{1} L<1+\beta \mathcal{D}^{i, o}\left(\lambda-\mu^{F}\right)$ and that $\left.\max _{L \in\left(L^{\text {ind }, \bar{L}]}\right.} \frac{d B(L)}{d L}-\left(\zeta_{0}+\zeta_{1} L\right)<0 \cdot\right]^{3}$ In this case, by Lemma C.4, the equilibrium intensity of violence is unique and is the solution to $\zeta_{0}+\zeta_{1} L=1+\beta \mathcal{D}^{i, o}\left(\lambda-\mu^{F}\right)$ if it exists or $L=L^{\text {ind }}$ otherwise.
Case 3. Suppose at $L=L^{\text {ind }}, \zeta_{0}+\zeta_{1} L<\max _{L \in\left(L^{i n d}, \bar{L}\right]} \frac{d B(L)}{d L}$. The equilibrium intensity is $L=L^{\text {ind }}$ if the equation $\zeta_{0}+\zeta_{1} L=\frac{d B(L)}{d L}$, with $\frac{d B(L)}{d L}$ defined by Equation C.12, has no solution. The equilibrium intensity is the solution to $\zeta_{0}+\zeta_{1} L=\frac{d B(L)}{d L}$ if it is unique. It is either the smallest solution to $\zeta_{0}+\zeta_{1} L=\frac{d B(L)}{d L}$ or $L=\bar{L}$ if there are multiple solutions.
To complete the proof, recall that $\kappa_{S}^{*}\left(L^{i n d}\right)=0, \bar{e}\left(1, L^{i n d}\right)=\bar{v}+\overline{V_{2}}+L$ Equation 5), and $\frac{d \bar{e}}{d L}=$ $\left(1-\kappa_{S}^{*}(L)\right)-\frac{\partial \kappa_{S}^{*}(L)}{\partial L}\left(\bar{v}+\overline{V_{2}}+L\right)$. Hence, $\lim _{L \downarrow L^{i n d}} \frac{d B(L)}{d L}=1-\frac{\partial \kappa_{S}^{*}(L)}{\partial L}\left(\bar{v}+\overline{V_{2}}+L\right)+\beta \frac{d \mu^{S}}{d L}\left(\bar{v}+\overline{V_{2}}+L\right) \mathcal{D}^{c, n c}$ so the condition of point (ii) of the proposition is is contained in Case 3 and $L^{*}>L^{i n d}$.

Lemma C.5. Assume $L^{\text {ind }}<\bar{L}$. If $C_{1} \leq \frac{1}{2(1-\lambda)\left(v(S, c)-v(S, n c)+V_{2}(c)-V_{2}(n c)\right)}$ the marginal benefit of violence is strictly positive for $L>L^{\text {ind }}$.

Proof. From the proof of Proposition 2, recall that

$$
\frac{d \bar{e}}{d L}>\frac{\beta \lambda \mathcal{D}^{c, n c}}{2 C_{1} \bar{e}}
$$

Using Equation C. 12 and $\mu_{\kappa_{F}}^{S}=0$, this implies that

$$
\begin{aligned}
\frac{d B(L)}{d L} & >\frac{\beta \lambda \mathcal{D}^{c, n c}}{2 C_{1} \bar{e}}+\beta \mu_{L}^{S} \mathcal{D}^{c, n c} \bar{e}\left(1-\kappa_{S}\right) \\
& =\frac{\beta \lambda \mathcal{D}^{c, n c}}{C_{1} \bar{e}}\left(\frac{1}{2}-C_{1}(1-\lambda)\left(v(S, c)-v(S, n c)+V_{2}(c)-V_{2}(n c)\right)\left(1-\kappa_{S}\right)^{3}\right)
\end{aligned}
$$

The second line uses: $\mu_{L}^{S}=-\lambda(1-\lambda) \frac{(v(S, c)-v(S, n c))+\left(V_{2}(c)-V_{2}(n c)\right)}{\left(\bar{v}+\overline{V_{2}}+L\right)^{2}}$ and $\bar{e}=\left(1-\kappa_{S}\right)\left(\bar{v}+\overline{V_{2}}+L\right)$. Since $\kappa_{S}^{*}(L) \geq 0$ for all $L \geq L^{\text {ind }}$, if $C_{1} \leq \frac{1}{2(1-\lambda)\left(v(S, c)-v(S, n c)+V_{2}(c)-V_{2}(n c)\right)}$, then $d B(L) / d L>0$.

## Proof of Corollary 2

We provide sufficient (and some necessary) conditions for a purge to be semi-discriminate.
Denote $\underline{r}:=\mu^{S}(0, \bar{L})$. Using the proof of Proposition 3 and $\mu^{S}(\cdot)$ decreasing with $L$ (Lemma

[^2]B. 1 and constant in $\kappa_{S}$, if $r \leq \underline{r}, L^{i n d}=\bar{L}$ for all $C_{0}, C_{1}$ and a purge is never semi-indiscriminate. So $r>\underline{r}$ is a necessary condition. This is condition 1 .

Supposing condition 1. holds, define $\overline{C_{0}}(r)=\beta\left[r-\mu^{S}(0, \bar{L})\right] \mathcal{D}^{c, n c}$. If $C_{0} \geq \overline{C_{0}}(r)$ then for all $C_{1}>0$, the purge cannot be semi-indiscriminate as the marginal cost is always greater than the marginal benefit. When $C_{0}<\overline{C_{0}}(r)$, define $\check{C}_{1}\left(r, C_{0}\right)$ such that at $L=\bar{L}, K_{S D}(0, L)=$ $\beta\left[r-\mu^{S}(0, L)\right] \mathcal{D}^{c, n c}-\left(C_{0}+C_{1}(1-\bar{e}(0, L))\right)=0$. Similarly, if $C_{1} \geq \check{C}_{1}\left(r, C_{0}\right)$, a purge can never be semi-indiscriminate. If $d B(L) / d L>0$ for all $L \leq L^{\text {ind }}$ at $C_{1}=\check{C}_{1}\left(r, C_{0}\right)$, then denote $\overline{C_{1}}\left(r, C_{0}\right):=\check{C}_{1}\left(r_{i}, c_{0}\right)$. If not, denote $\overline{C_{1}}\left(r, C_{0}\right)$, the smallest $C_{1}$ such that for all $C_{1}<\overline{C_{1}}\left(r, C_{0}\right)$, the marginal benefit satisfies $B(L)>0$ for all $L \geq L^{\text {ind }}$ (such $\overline{C_{1}}$ exists by Lemma C.5). This is condition 2.4
Finally define $\overline{\zeta_{0}}(r):=\frac{d B\left(L^{i n d}\right)}{d L}$ (condition 2 ensures $\frac{d B(L)}{d L}>0$ over this range). And for all $\zeta_{0} \leq \overline{\zeta_{0}}(r)$, denote $\overline{\zeta_{1}}\left(r, \zeta_{0}\right):=\frac{\frac{d B(L)}{d L}-\zeta_{0}}{L}$ at $L=L^{\text {ind }}$. This guarantees that for all $\zeta_{0}<\overline{\zeta_{0}}(r)$ and $\zeta_{1}<\overline{\zeta_{1}}\left(r, \zeta_{0}\right)$, the condition described in point (ii) of Proposition 5 holds and the purge is semi-discriminate. This is condition 3.

## Proof of Proposition 6

The procedure is as such. Step 1: Pick $\left(\lambda^{\prime}, r^{\prime}, v(S, c)^{\prime}, v(S, n c)^{\prime}, C_{0}^{\prime}, \zeta_{0}^{\prime}\right) \in[0,1]^{3} \times[0, v(S, c)] \times \mathbb{R}_{+}^{2}$. Step 2: Check whether there exists $C_{1}^{d}$ satisfying Equation 8 and $\zeta_{1}^{d} \in \mathbb{R}_{+}$such that (i) there exists a local maximum of $B(L)-\zeta(L)$ in $\left[0, L^{\text {full }}\right]$, denoted $L^{1}$ as in Proposition 5 and (ii) $B\left(L^{1}\right)=B\left(L^{\text {full }}\right)$ (notice that $C_{1}^{d}$ and $\zeta_{1}^{d}$ are unique if they exist). Step 3: If conditions (i) and (ii) hold then $\left(\lambda^{\prime}, r^{\prime}, v(S, c)^{\prime}, v(S, n c)^{\prime}, C_{0}^{\prime}, \zeta_{0}^{\prime}\right) \in \mathcal{P}^{d}$, if not $\left(\lambda^{\prime}, r^{\prime}, v(S, c)^{\prime}, v(S, n c)^{\prime}, C_{0}^{\prime}, \zeta_{0}^{\prime}\right) \notin \mathcal{P}^{d}$. Repeat the steps for all possible $\left(\lambda, r, v(S, c), v(S, n c), C_{0}, \zeta_{0}\right) . \mathcal{P}^{d}$ is non-empty as we can always pick $C_{1}$ such that a fully discriminate purge is possible and $\zeta_{0}$ and $\zeta_{1}$ such that conditions (i) and (ii) hold by convexity of the marginal benefit (Lemma C.1). $\mathcal{P}^{d}$ is not measure 0 as we can always perturb the parameters slightly and adjust $\zeta_{0}$ and $\zeta_{1}$. Due to the convexity of the marginal benefit of violence and conditions (i) and (ii), the claim holds directly ${ }^{5}$

[^3]Remark C.1. Suppose Equation C. 13 does not hold. There exists a generically unique equilibrium intensity of violence.

Proof. We amend the proof of Proposition 5 to take into account the discontinuity at $L=L^{\text {full }}$ (Lemma C.3). First, note that $B(L)$ is always bounded so a maximum exists. Suppose there exists $L^{\prime}$ such that $\zeta_{0}+\zeta_{1} L^{\prime}=\frac{d B\left(L^{\prime}\right)}{d L}$ and $L^{\prime}<L^{\text {full }}$. If there exists $L^{\prime \prime}>L^{\text {full }}$ such that $\zeta_{0}+\zeta_{1} L^{\prime \prime}=\frac{d B\left(L^{\prime \prime}\right)}{d L}$, then the equilibrium intensity of violence satisfies $L^{*}=\arg \max _{L \in\left\{L^{\prime}, L^{\prime \prime}\right\}} B(L)$ and is generically unique. If there is no such $L^{\prime \prime}$, then the equilibrium intensity of violence satisfies $L^{*}=\arg \max _{L \in\left\{L^{\prime}, L^{f u l l}\right\}} B(L)$ and is generically unique.

## D Proofs of extensions (Section 6)

## D. 1 Endogenous reward

As explained in the text, the autocrat can supplement agents' second-period benefit with $R_{2}$ at marginal cost $\chi^{\prime}\left(R_{2}\right)=\chi_{0}+\chi_{1} R_{2}$ with $\chi_{0}=\zeta_{0}$ to simplify the analysis. In all this subsection, we denote equilibrium value by $\uparrow$. The previous analysis corresponds to the case when $R_{2}$ is constrained to be 0 . We also amend the notation of the baseline model and use $L^{*}(0)$ to denote the equilibrium intensity of violence characterized in Proposition 5.

In the setting with endogenous reward $R_{2}$, agents' efforts become:

$$
e_{1}^{i}(\tau)= \begin{cases}v(S, \tau)+\kappa_{F}\left(V_{2}(\tau)+L+R_{2}\right) & \text { if } \kappa_{S}=0  \tag{D.1}\\ \left(1-\kappa_{S}\right)\left(v(S, \tau)+V_{2}(\tau)+L+R_{2}\right) & \text { if } \kappa_{S}>0\end{cases}
$$

It is useful to denote $T=R_{2}+L$ and $\bar{T}=1-v(S, c)-V_{2}(c)$. We can rewrite effort as:
(i) $e_{1}^{i}\left(\kappa_{F}, T ; \tau\right)=v(S, \tau)+\kappa_{F}\left(V_{2}(\tau)+T\right)$ in a discriminate purge;
(ii) $e^{i}\left(\kappa_{S}, T ; \tau\right)=\left(1-\kappa_{S}\right)\left(v(S, \tau)+V_{2}(\tau)+T\right)$ in a semi-indiscriminate purge.

On the agents' side, the problems in the constrained ( $R_{2}=0$ ) and unconstrained ( $R_{2}$ endogenous) cases are isomorphic. The only difference is that $L$ is replaced by $T=R_{2}+L$. Hence, all the comparative statics above hold in this setting replacing $L$ by $T$. In particular, we recover the following results.
(i) There exist a unique $T^{\text {full }}=L^{\text {full }}$ and $T^{\text {ind }}=L^{\text {ind }}$ such that the purge is partially discriminate if and only if $T \leq T^{\text {full }}$, fully discriminate if and only if $T \in\left(T^{f u l l}, T^{\text {ind }}\right]$, semi-indiscriminate otherwise.
(ii) The expected benefit of $T$-denoted $B(T)$-is strictly increasing in $T$ for $T<T^{\text {ind }}$, strictly convex for $T \leq T^{\text {full }}$ and strictly concave for $T \in\left(T^{\text {full }}, T^{\text {ind }}\right)$.

We further assume that Equation 8 holds as well as a modified version (replacing $L^{\text {full }}$ by $T^{\text {full }}$ ) of Equation C.13. This last assumption implies that $T^{\text {full }}<\bar{T}$ and $B(T)$ is continuous (as before this last assumption only simplifies the analysis). Further, an appropriately modified Lemma C. 4 (with $T$ replacing $L$ ) holds.

The autocrat's problem can then be conceived into steps: 1 ) for all $T$, find $\widehat{L}(T)$ and $\widehat{R_{2}}(T)$ which minimizes the cost of producing $T$ and 2) Find the optimal $T$ given step 1. Regarding step 1, the autocrat's cost of producing $T$ is thus:

$$
\min _{L, R \in \mathbb{R}_{+}^{2}} \zeta(L)+\chi(T) \text { such that } L+R=T
$$

Ignoring the non-negativity constraint, the solution to the minimization problem is:

$$
\begin{aligned}
\widehat{L}(T) & =\frac{\chi_{0}-\zeta_{0}+\chi_{1}}{\chi_{1}+\zeta_{1}} T
\end{aligned}=\frac{\chi_{1}}{\chi_{1}+\zeta_{1}} T T+\zeta_{1} \chi_{1}+\zeta_{1} \quad T=\frac{\zeta_{1}}{\chi_{1}+\zeta_{1}} T T
$$

Hence, under our assumption that $\zeta_{0}=\chi_{0}$, the non-negativity constraint does not bind.
Denote now $\mathcal{T}(T):=\zeta(\widehat{L}(T))+\chi\left(\widehat{R_{2}}(T)\right)$. Observe that $\mathcal{T}(T)$ is strictly increasing and convex, $\mathcal{T}(T)<\zeta(T)$ for all $T$. Further $\mathcal{T}(T)$ satisfies:

$$
\begin{equation*}
\mathcal{T}^{\prime}(T)=\zeta_{0}+\zeta_{1} \frac{\chi_{1}}{\chi_{1}+\zeta_{1}} T=\zeta^{\prime}(\widehat{L}(T)) \tag{D.2}
\end{equation*}
$$

We can now prove Proposition 7 .

## Proof of Proposition 7

As a preliminary, we establish that $\widehat{T} \geq L^{*}(0)$.
Suppose first that $L^{*}(0) \notin\left\{L^{\text {full }}, L^{\text {ind }}, 1-v(S ; c)-V_{2}(c)\right\}$. Then it must be that $\zeta^{\prime}\left(L^{*}(0)\right)=$ $B^{\prime}\left(L^{*}(0)\right)$. Given $\mathcal{T}^{\prime}\left(L^{*}(0)\right)=\zeta^{\prime}\left(\widehat{L}\left(L^{*}(0)\right)<\zeta^{\prime}\left(L^{*}(0)\right)\right.$, we necessarily have $\mathcal{T}^{\prime}\left(L^{*}(0)\right)<B^{\prime}\left(L^{*}(0)\right)$. Given $L^{*}(0) \notin\left\{L^{\text {full }}, L^{\text {ind }}, 1-v(S ; c)-V_{2}(c)\right\}$, there exists $\eta>0$ such that $\mathcal{T}^{\prime}\left(L^{*}(0)+\eta\right)<$ $B^{\prime}\left(L^{*}(0)+\eta\right)$ which implies $\widehat{T}>L^{*}(0)$.
If $L^{*}(0)=L^{\text {full }}$, then there are two cases to consider: (a) $\lim _{T \downarrow T^{\text {full }}} B^{\prime}(T) \leq \zeta^{\prime}\left(\widehat{L}\left(T^{\text {full }}\right)\right.$ then $\widehat{T}=$ $T^{\text {full }}=L^{\text {full }}$ and (b) $\lim _{T \downarrow T^{\text {full }}} B^{\prime}(T)>\zeta^{\prime}\left(\widehat{L}\left(T^{f u l l}\right)\right.$ then $\widehat{T}>L^{*}(0)$. A similar reasoning holds for
$L^{*}(0)=L^{\text {ind }}$. If $L^{*}(0)=1-v(S ; c)-V_{2}(c)$, given the lower marginal cost of producing $\widehat{T}$, we necessarily have $\widehat{T}=L^{*}(0)$ then.
Using this result, we can now prove points (i)-(iii).
Point (i). Purge inference is weakly increasing in $L$ (proof of Proposition 1) in the baseline model. Hence, it is weakly increasing in $T$. Noting that since the agents' problem is isomorphic, Lemma 3 holds in this extension so the purge inference is the same in both settings whenever $T=L$ (i.e., $\left.\widehat{\kappa}_{\omega}(L)=\kappa_{\omega}^{*}(L), \omega \in\{F, S\}\right)$. Since $\widehat{T} \geq L^{*}(0)$ we have: $\widehat{\kappa}_{F}(\widehat{T}) \geq \kappa_{F}^{*}\left(L^{*}(0)\right)$ and $\widehat{\kappa}_{S}(\widehat{T}) \geq \kappa_{S}^{*}\left(L^{*}(0)\right)$ with equality only if $L^{*}(0), \widehat{T} \in\left\{L^{f u l l}, L^{\text {ind }}, 1-v(S ; c)-V_{2}(c)\right\}^{2}$.
Point (ii). By a similar reasoning as above, whenever $T=L$, the purge breadth is the same in the extension as in the baseline model (i.e., $\widehat{\kappa}(L)=\kappa^{*}(L)$ ). The purge breadth is strictly increasing for $L \in\left[0, L^{\text {full }}\right]$ and strictly decreasing for $L \in\left[L^{\text {full }}, L^{\text {ind }}\right]$. Suppose $L^{*}(0), \widehat{T} \in\left(0, L^{\text {full }}\right)^{2}{ }^{6}$ Since $\widehat{T}>L^{*}(0)$ by the reasoning above, $\kappa^{*}\left(L^{*}(0)\right)<\widehat{\kappa}(\widehat{T})$ then. Suppose $L^{*}(0), \widehat{T} \in\left(L^{\text {full }}, L^{\text {ind }}\right)^{2}$. Since $\widehat{T}>L^{*}(0)$ by the reasoning above, $\kappa^{*}\left(L^{*}(0)\right)>\widehat{\kappa}(\widehat{T})$ then.
Point (iii). Suppose $\widehat{T}<T^{\text {full }}$ and so is $L^{*}(0)$. We then have: $B^{\prime}(\widehat{T})=\mathcal{T}^{\prime}(\widehat{T})=\zeta^{\prime}(\widehat{L}(\widehat{T}))$ and $B^{\prime}\left(L^{*}(0)\right)=\zeta^{\prime}\left(L^{*}(0)\right)$. Hence, $\widehat{L}(\widehat{T})=\left(\zeta^{\prime}\right)^{-1}\left(B^{\prime}(\widehat{T})\right)$ and $L^{*}(0)=\left(\zeta^{\prime}\right)^{-1}\left(B^{\prime}\left(L^{*}(0)\right)\right)$. Since $B^{\prime}(\cdot)$ is increasing (Lemma C.1) and $\widehat{T}>L^{*}(0)$ in that range (see above), $\widehat{L}(\widehat{T})>L^{*}(0)$. Suppose $\widehat{T}=L^{*}(0)=L^{\text {full }}$. Then $\widehat{L}(\widehat{T})=\frac{\xi_{1}}{\xi_{1}+\zeta_{1}} \widehat{T}<L^{*}(0)$ (the result holds for other parameter values). Hence, the equilibrium intensity of violence can be greater or lower.

## D. 2 Declining replacement pool

Recall that the purge breadth is $\kappa=\alpha_{F} \kappa_{F}+\alpha_{S} \kappa_{S}$. Suppose that the replacement pool is linearly decreasing in the purge breadth: $r(\kappa)=\bar{r}-r_{1} \kappa$. In a partially discriminate purge, the autocrat then maximizes with respect to $\kappa_{F}$ :

$$
\left(\int_{0}^{\kappa_{F}\left(1-\bar{e}_{1}\right)} r(z) d z-\mu^{F} \kappa_{F}\left(1-\bar{e}_{1}\right)\right) \mathcal{D}^{c, n c}
$$

Rearranging, this is equivalent to

$$
\kappa_{F}\left(1-\bar{e}_{1}\right)\left(\bar{r}-\mu^{F}\right) \mathcal{D}^{c, n c}-r_{1} \mathcal{D}^{c, n c} \frac{\left(\left(1-\bar{e}_{1}\right) \kappa_{F}\right)^{2}}{2}
$$

The autocrat's problem is then as in the baseline model with $r=\bar{r}, C_{0}=0$ and $C_{1}=r_{1} \mathcal{D}^{c, n c}$. A similar mapping exists for a semi-indiscriminate purge. Hence, we can apply the same reasoning as in Appendices $A-C$ and show that all our results hold in this setting.

[^4]
## D. 3 Autocrat's survival

In this extension, we suppose that the autocrat cares about staying in power and gets a payoff of 1 if so. The probability the autocrat survives is:

$$
\begin{equation*}
P(\text { survives })=\gamma \bar{e}_{1}+(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right) \times \beta(1-\mathcal{P}(L)) \tag{D.3}
\end{equation*}
$$

with $\mathcal{P}(L)$ is the proportion of congruent types among second period subordinate (see the proof of Proposition (4). Observe that $\epsilon$ measures the complementarity between first-period performance and the proportion of congruent surbordinates.

Our first Lemma reproduces Lemma 3 in this framework. To do so, define $\widehat{\mathcal{P}}(L):=\hat{\alpha}_{F}(L) r+$ $\lambda\left(v(S, c)+V_{2}(c)+L\right)\left(\right.$ recall $\left.\hat{\alpha}_{F}(L)=1-\bar{v}-\overline{V_{2}}-L\right)$ and $\widehat{\mu^{F}}(L)=\lambda \frac{1-v(S, c)-V_{2}(c)-L}{\hat{\alpha}_{F}(L)}$. Using $\mu^{S}(L)=\lambda \frac{v(S, c)+V_{2}(c)+L}{\bar{v}+\overline{V_{2}}+L}$, we obtain:

Lemma D.1. For each $L$, there exists unique equilibrium purge incidences $\kappa_{F}^{*}(L), \kappa_{S}^{*}(L)$.
Further, there exists $\bar{\epsilon}^{\kappa}(L)>0$ such that if $\epsilon \leq \bar{\epsilon}^{\kappa}(L)$, then whenever
(i) $C_{0}+C_{1} \times \hat{\alpha}_{F}(L)>-(1-\gamma)\left((1-\epsilon)+\hat{\alpha}_{F}(L) \epsilon\right)\left(r-\widehat{\mu^{F}}(L)\right) \beta^{\prime}(1-\widehat{\mathcal{P}}(L))$, the purge is partially discriminate;
(ii) $C_{0}+C_{1} \times \hat{\alpha}_{F}(L)<-(1-\gamma)\left((1-\epsilon)+\hat{\alpha}_{F}(L) \epsilon\right)\left(r-\mu^{S}(L)\right) \beta^{\prime}(1-\widehat{\mathcal{P}}(L))$, the purge is semiindiscriminate;
(iii) The purge is fully discriminate otherwise.

Proof. From Equation B.9, if $\kappa_{F} \in(0,1), \mathcal{P}(L)=\left(1-\bar{e}_{1}\right) \kappa_{F}\left(r-\mu^{F}\right)+\lambda$ (ignoring arguments whenever possible), with $\bar{e}_{1}=\bar{v}+\kappa_{F}\left(\overline{V_{2}}+L\right)$ since the agents' problem is unchanged. Define

$$
\begin{equation*}
S_{P D}\left(\kappa_{F}, L\right)=-(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right)\left(r-\mu^{F}\right) \beta^{\prime}(1-\mathcal{P}(L))-C_{0}-C_{1} \kappa_{F}\left(1-\bar{e}_{1}\right) \tag{D.4}
\end{equation*}
$$

If the purge is partially discriminate, $\kappa_{F}^{*}(L)$ is defined as a solution to $S_{P D}\left(\kappa_{F}, L\right)=0$ since the autocrat takes effort and violence as given at the time of her purging decision. Notice that for $\kappa_{F}=1$, we obtain: $S_{P D}(1, L)=-(1-\gamma)\left((1-\epsilon)+\hat{\alpha}_{F}(L) \epsilon\right)\left(r-\widehat{\mu^{F}}(L)\right) \beta^{\prime}(1-\widehat{\mathcal{P}}(L))-C_{0}-C_{1} \times \hat{\alpha}_{F}(L)$. From Equation B.12, if $\kappa_{S} \in(0,1), \mathcal{P}(L)=r-\bar{e}_{1}\left(1-\kappa_{S}\right)\left(r-\mu^{S}\right)$, with $\bar{e}_{1}=\left(1-\kappa_{S}\right)\left(\bar{v}+\overline{V_{2}}+L\right)$. Define

$$
\begin{equation*}
S_{S D}\left(\kappa_{S}, L\right)=-(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right)\left(r-\mu^{S}\right) \beta^{\prime}(1-\mathcal{P}(L))-C_{0}-C_{1}\left(1-\left(1-\kappa_{S}\right) \bar{e}_{1}\right) \tag{D.5}
\end{equation*}
$$

If the purge is partially discriminate, $\kappa_{S}^{*}(L)$ is defined as a solution to $S_{S D}\left(\kappa_{S}, L\right)=0$ since the autocrat takes effort and violence as given at the time of her purging decision. Notice that for
$\kappa_{S}=0$, we obtain: $S_{S D}(0, L)=-(1-\gamma)\left((1-\epsilon)+\hat{\alpha}_{F}(L) \epsilon\right)\left(r-\mu^{S}(L)\right) \beta^{\prime}(1-\widehat{\mathcal{P}}(L))-C_{0}-C_{1} \times$ $\hat{\alpha}_{F}(L)<S_{P D}(1, L)$ since $\mu^{S}>\widehat{\mu^{F}}$. We now show that $\partial S_{S D}\left(\kappa_{S}, L\right) / \partial \kappa_{S}<0$ for $\epsilon$ not too large. Using the definition of $\mathcal{P}(L)$ and since $\mu^{S}$ does not depend on $\kappa_{S}$, we obtain:

$$
\begin{align*}
\frac{\partial S_{S D}\left(\kappa_{S}, L\right)}{\partial \kappa_{S}}= & -(1-\gamma) \epsilon\left(\bar{v}+\overline{V_{2}}+L\right)\left(r-\mu^{S}\right) \beta^{\prime}(1-\mathcal{P}(L)) \\
& +(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right)\left(r-\mu^{S}\right)^{2} 2 \bar{e}_{1} \beta^{\prime \prime}(1-\mathcal{P}(L))-2 C_{1} \bar{e}_{1} \tag{D.6}
\end{align*}
$$

Notice that under the assumption that $\beta^{\prime \prime}(\cdot) \leq 0$, the terms on the second line are both negative. Hence, when $\epsilon=0, \frac{\partial S_{S D}\left(\kappa_{S}, L\right)}{\partial \kappa_{S}}<0$. Since $\frac{\partial S_{S D}\left(\kappa_{S}, L\right)}{\partial \kappa_{S}}$ is continuous in $\epsilon$, there exists $\bar{\epsilon}^{\kappa}(L)$ such that for all $\epsilon \leq \bar{\epsilon}^{\kappa}(L), \frac{\partial S_{S D}\left(\kappa_{S}, L\right)}{\partial \kappa_{S}} \leq 0$.
Hence, if $\epsilon \leq \bar{\epsilon}^{\kappa}(L)$, the properties of $S_{S D}\left(\kappa_{S}, L\right)$ yield $S_{S D}\left(\kappa_{S}, L\right)<S_{P D}(1, L)$ for all $\kappa_{S} \geq 0$. We can then apply a similar reasoning as in the proof of Lemma 3 to prove the claim. Note, however, that there may be multiple solutions to $S_{P D}\left(\kappa_{F}, L\right)=0$ even if point (i) holds. In this case, we select the highest solution.

Our next two Lemmas establish that the purge incidence $\kappa_{\omega}^{*}(L)$ and purge breadth $\kappa^{*}(L)$ are strictly increasing with $L$ in a partially discriminate purge $(\omega=F)$ and semi-indiscriminate purge ( $\omega=S$ ) when $\epsilon$ is not too large.

Lemma D.2. If $r \geq \lambda$, there exists $\bar{\epsilon}^{P D}(L)>0$ such that if $\epsilon<\bar{\epsilon}^{P D}(L)$, then in a partially discriminate purge, the purge inference $\kappa_{F}^{*}(L)$ and breadth $\kappa^{*}(L)$ are strictly increasing with $L$.

Proof. Since we select the highest purge inference, by a similar reasoning as in the proof of Proposition 1. $\partial S_{P D}\left(\kappa_{F}^{*}(L), L\right) / \partial \kappa_{F}<0$ in a partially discriminate purge (i.e., condition (i) in Lemma D. 1 holds). We thus just need to show that $\partial S_{P D}\left(\kappa_{F}^{*}(L), L\right) / \partial L>0$ to prove that $\kappa_{F}^{*}(L)>0$ (by the Implicit Function Theorem). Observe that (using subscript to denote partial derivative):

$$
\begin{align*}
\frac{\partial S_{P D}\left(\kappa_{F}^{*}(L), L\right)}{\partial L}= & \epsilon \kappa_{F}^{*}(L) \beta^{\prime}(1-\mathcal{P}(L)) \\
& +(1-\gamma)\left(1-\epsilon+\epsilon\left(1-\bar{e}_{1}\right)\right) \mu_{L}^{F} \beta^{\prime}(1-\mathcal{P}(L)) \\
& -(1-\gamma)\left(1-\epsilon+\epsilon\left(1-\bar{e}_{1}\right)\right)\left(r-\mu^{F}\right) \kappa_{F}^{*}(L)^{2}(r-\lambda) \beta^{\prime \prime}(1-\mathcal{P}(L)) \\
& +C_{1} \kappa_{F}^{*}(L)^{2} \tag{D.7}
\end{align*}
$$

Under the assumptions, given $\mu_{L}^{F}<0$ (see Lemma B.1), the terms on the last three lines are strictly positive. Hence, when $\epsilon=0, \frac{\partial S_{P D}\left(\kappa_{F}^{*}(L), L\right)}{\partial L}>0$. Since $\frac{\partial S_{P D}\left(\kappa_{F}^{*}(L), L\right)}{\partial L}$ is continuous in $\epsilon$, there exists $\bar{\epsilon}_{1}^{P D}(L)>0$ (possibly equals 1 ) such that for all $\epsilon<\bar{\epsilon}_{1}^{P D}(L), \frac{\partial S_{P D}\left(\kappa_{F}^{*}(L), L\right)}{\partial L}>0$ and the purge
inference is strictly increasing with $L$.
Further, recall that the purge breadth is $\left(1-\bar{e}_{1}\right) \kappa_{F}$ so the purge breadth is strictly increasing with $L$ whenever $\frac{\partial S_{P D}\left(\kappa_{F}^{*}(L), L\right)}{\partial L}-C_{1} \kappa_{F}^{*}(L)^{2}$ is strictly positive. This is guaranteed for $\epsilon=0$ using Equation D. 7 and the assumptions on $\beta(\cdot)$. Hence, there exists $\bar{\epsilon}_{2}^{S D}(L)>0$ (possibly equals 1) such that for all $\epsilon<\bar{\epsilon}_{2}^{S D}(L)$, the purge breadth is strictly increasing with $L$. Since $C_{1} \kappa_{F}^{*}(L)^{2}>0$, $\bar{\epsilon}_{2}^{P D} \leq \bar{\epsilon}_{1}^{P D}$ (with strict inequality whenever $\bar{\epsilon}_{2}^{P D}(L)<1$ ). Denote $\bar{\epsilon}^{P D}(L):=\bar{\epsilon}_{2}^{P D}(L)$ so that the claim holds.

Lemma D.3. There exists $\bar{\epsilon}^{S D}(L)>0$ such that if $\epsilon<\bar{\epsilon}^{S D}$, then in a partially discriminate purge, the purge inference $\kappa_{S}^{*}(L)$ and breadth $\kappa^{*}(L)$ are strictly increasing with $L$.

Proof. The proof proceeds along the same lines as the proof of Lemma D. 3 noting that a necessary condition for the purge to be semi-indiscriminate is $r>\lambda$ and given Equation D. 5

$$
\begin{align*}
\frac{\partial S_{S D}\left(\kappa_{S}^{*}(L), L\right)}{\partial L}= & (1-\gamma) \epsilon\left(1-\kappa_{S}^{*}(L)\right) \beta^{\prime}(1-\mathcal{P}(L)) \\
& +(1-\gamma)\left(1-\epsilon+\epsilon\left(1-\bar{e}_{1}\right)\right) \mu_{L}^{S} \beta^{\prime}(1-\mathcal{P}(L)) \\
& -(1-\gamma)\left(1-\epsilon+\epsilon\left(1-\bar{e}_{1}\right)\right)\left(r-\mu^{S}\right)\left(1-\kappa_{S}^{*}(L)\right)^{2}(r-\lambda) \beta^{\prime \prime}(1-\mathcal{P}(L)) \\
& +C_{1}\left(1-\kappa_{S}^{*}(L)\right)^{2} \tag{D.8}
\end{align*}
$$

with $\mu_{L}^{S}<0$.
The following condition is the equivalent to Equation 8 in this setting

$$
\begin{equation*}
C_{0}+C_{1} \hat{\alpha}_{F}(\bar{L})<-(1-\gamma)\left((1-\epsilon)+\epsilon \hat{\alpha}_{F}(\bar{L})\right) \beta^{\prime}\left(1-\lambda-r \hat{\alpha}_{F}(\bar{L})\right) \tag{D.9}
\end{equation*}
$$

We can now state the equivalent to Proposition 1 and in this setting.
Proposition D.1. If $r \geq \lambda$, there exists $\bar{\epsilon}>0$ such that if $\epsilon<\bar{\epsilon}$, then

1. If Equation D.9 does not hold, then for all intensity of violence, the purge is partially discriminate: $\kappa_{F}^{*}(L) \in[0,1)$.
2. If Equation D.9 holds, then there exist unique $L^{f u l l}<\bar{L}$ and $L^{\text {ind }} \in\left(L^{f u l l}, \bar{L}\right]$ such that:
(i) For $L<L^{\text {full }}$, the purge is partially discriminate $\left(\kappa_{F}^{*}(L) \in[0,1)\right)$;
(ii) For $L \in\left[L^{\text {full }}, L^{\text {ind }}\right]$, the purge is fully discriminate $\left(\kappa_{F}^{*}(L)=1\right.$ and $\left.\kappa_{S}^{*}(L)=0\right)$;
(iii) For $L>L^{\text {ind }}$, the purge is semi-indiscriminate $\left(\kappa_{S}^{*}(L)>0\right)$.

Proof. Denote $\bar{\epsilon}=\min _{L \in[0, \bar{L}]} \min \left\{\bar{\epsilon}^{\kappa}(L), \bar{\epsilon}^{P D}(L), \bar{\epsilon}^{S D}(L)\right\}>0$. For all $\epsilon<\bar{\epsilon}$, we can then apply the same reasoning as in the proof of Proposition 1 using Lemmas D.1.D.3.

Proposition D.2. If $r \geq \lambda$ and $\epsilon<\bar{\epsilon}$ (with $\bar{\epsilon}$ defined in the text of Proposition D.1), then the relationship between the purge breadth $\kappa^{*}(L)$ and the intensity of violence $L$ exhibits the following properties:
(i) For $L<L^{\text {full }}, \kappa^{*}(L)$ is strictly increasing in $L$;
(ii) For $L \in\left[L^{\text {full }}, L^{\text {ind }}\right], \kappa^{*}(L)$ is strictly decreasing in $L$;
(iii) For $L>L^{\text {ind }}, \kappa^{*}(L)$ is strictly increasing in $L$.

Proof. Follows directly from Lemmas D. 2 and D. 3 and Proposition D. 1.
The next proposition establishes that the fear effect holds in this setting as long as the survival probability is not too concave.

Proposition D.3. There exists $\underline{\beta}<0$ such that if $r \geq \lambda, \epsilon<\bar{\epsilon}$ (with $\bar{\epsilon}$ defined in the text of Proposition D.1 and $\min _{z \in[1-r, 1]} \beta^{\prime \prime}(z)>\underline{\beta}$, then:

1. The total derivative of effort with respect to violence $\frac{d \bar{e}(L)}{d L}$ is always strictly positive.
2. Further, there exists $L^{\text {fear }} \leq L^{\text {full }}$ such that the derivative satisfies:
(i) $\frac{d \bar{e}(L)}{d L}>1$ for all $L \in\left(L^{\text {fear }}, L^{\text {full }}\right)$;
(ii) $\frac{d \bar{e}(L)}{d L}=1$ for all $L \in\left[L^{\text {full }}, L^{\text {ind }}\right)$;
(iii) $\frac{d \bar{e}(L)}{d L}<1$ for all $L \geq L^{\text {ind }}$.

Proof. We consider the three types of purges in turn. First, in a partially discriminate purge, equilibrium first-period performance is (ignoring superscript and subscript):

$$
\bar{e}(L)=\bar{v}+\kappa_{F}^{*}(L)\left(\overline{V_{2}}+L\right)
$$

As in the baseline model, we obtain: $\frac{d \bar{e}(L)}{d L}=\kappa_{F}^{*}(L)+\frac{\partial \kappa_{F}^{*}(L)}{\partial L}\left(\overline{V_{2}}+L\right)>0$ since $\frac{\partial \kappa_{F}^{*}(L)}{\partial L}>0$ under the assumption that $\epsilon<\bar{\epsilon}$ (Lemma D.2). Given that $\kappa_{F}^{*}(L)$ is not necessarily continuous, we need to consider two cases. Case 1: there is an intensity of violence $L$ such that $\kappa_{F}^{*}(L)+\frac{\partial \kappa_{F}^{*}(L)}{\partial L}\left(\overline{V_{2}}+L\right)>1$ (this is the case if $\kappa_{F}^{*}(L)$ is continuous in $L$ ). In this case, denote $L^{\prime}=\max \left\{L: \kappa_{F}^{*}(L)+\frac{\partial \kappa_{F}^{*}(L)}{\partial L}\left(\overline{V_{2}}+\right.\right.$
 $L^{\text {fear }}=L^{\text {full }}$. Case 2: there is no $L$ such that $\kappa_{F}^{*}(L)+\frac{\partial \kappa_{F}^{*}(L)}{\partial L}\left(\overline{V_{2}}+L\right)>1$. In this case denote $L^{\text {fear }}:=L^{\text {full }}$.

Let us now turn to a fully discriminate purge. In this case, the equilibrium first-performance is:

$$
\bar{e}(L)=\bar{v}+\overline{V_{2}}+L
$$

[^5]and $\frac{d e(L)}{d L}=1$ as claimed.
Finally, in a semi-indiscriminate purge, the equilibrium first-period performance is:
$$
\bar{e}(L)=\left(1-\kappa_{S}^{*}(L)\right)\left(\bar{v}+\overline{V_{2}}+L\right)
$$

Using $S_{S D}\left(\kappa_{S}^{*}(L), L\right)=0$, we obtain (ignoring all superscripts and arguments whenever possible)

$$
\frac{d \bar{e}(L)}{d L}=\left(1-\kappa_{S}\right)+\frac{\frac{\partial S_{S D}\left(\kappa_{S}(L), L\right)}{\partial L}}{\frac{\partial S_{S D}\left(\kappa_{S}(L), L\right)}{\partial \kappa_{S}}}\left(\bar{v}+\overline{V_{2}}+L\right)
$$

Given $\frac{\partial S_{S D}\left(\kappa_{S}(L), L\right)}{\partial \kappa_{S}}<0$ (since we select the highest purge inference) and $\frac{\partial S_{S D}\left(\kappa_{S}(L), L\right)}{\partial L}>0$ (Lemma D.3), clearly $\frac{d \bar{e}(L)}{d L}<1$. Further, $\frac{d \bar{e}(L)}{d L}$ has opposite sign than (using Equation D. 6 and Equation D. 8 .

$$
\begin{aligned}
S D= & -(1-\gamma) \epsilon \bar{e}\left(r-\mu^{S}\right) \beta^{\prime}(1-\mathcal{P}(L))-2 C_{1} \bar{e}_{1}\left(1-\kappa_{S}\right) \\
& +(1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e}))\left(r-\mu^{S}\right)^{2} 2 \bar{e}\left(1-\kappa_{S}\right) \beta^{\prime \prime}(1-\mathcal{P}(L)) \\
& +\left((1-\gamma) \epsilon \bar{e}\left(r-\mu^{S}\right) \beta^{\prime}(1-\mathcal{P}(L))+(1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e})) \mu_{L}^{S}\left(\bar{v}+\overline{V_{2}}+L\right) \beta^{\prime}(1-\mathcal{P}(L))\right. \\
& \left.-(1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e}))\left(r-\mu^{S}\right) \bar{e}\left(1-\kappa_{S}\right)(r-\lambda) \beta^{\prime \prime}(1-\mathcal{P}(L))+C_{1} \bar{e}\left(1-\kappa_{S}\right)\right) \\
= & -C_{1} \bar{e}\left(1-\kappa_{S}\right)+(1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e})) \mu_{L}^{S}\left(\bar{v}+\overline{V_{2}}+L\right) \beta^{\prime}(1-\mathcal{P}(L)) \\
& +(1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e}))\left(r-\mu^{S}\right) \bar{e}\left(1-\kappa_{S}\right) \beta^{\prime \prime}(1-\mathcal{P}(L))\left(2\left(r-\mu^{S}\right)-(r-\lambda)\right)
\end{aligned}
$$

$\operatorname{Using} S_{S D}\left(\kappa_{S}^{*}(L), L\right)=0 \Leftrightarrow C_{0}+C_{1}-C_{1} \bar{e}\left(1-\kappa_{S}^{*}(L)\right)=-(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right)\left(r-\mu^{S}\right) \beta^{\prime}(1-$ $\mathcal{P}(L)$ ), we obtain:

$$
\begin{aligned}
S D= & (1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e})) \beta^{\prime}(1-\mathcal{P}(L))\left(\mu_{L}^{S}\left(\bar{v}+\overline{V_{2}}+L\right)-\left(r-\mu^{S}\right)\right)-C_{0}-C_{1} \\
& +(1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e}))\left(r-\mu^{S}\right) \bar{e}\left(1-\kappa_{S}\right) \beta^{\prime \prime}(1-\mathcal{P}(L))\left(r+\lambda-2 \mu^{S}\right)
\end{aligned}
$$

Given $\mu^{S}=\lambda \frac{v(S, c)+V_{2}(c)+L}{\bar{v}+\overline{V_{2}}+L}$, we obtain $\left(\bar{v}+\overline{V_{2}}+L\right) \mu_{L}^{S}=-\lambda(1-\lambda) \frac{v(S, c)-v(S, n c)+V_{2}(c)-V_{2}(n c)}{\bar{v}+\overline{V_{2}}+L}=\lambda-\mu^{S}$. Further, using the assumption $C_{0}+C_{1}>-(1-\gamma) r \beta^{\prime}(1-r)>-(1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e})) r \beta^{\prime}(1-r)$, we have:

$$
\begin{gathered}
S D<(1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e}))\left((\lambda-r) \beta^{\prime}(1-\mathcal{P}(L))+r \beta^{\prime}(1-r)\right. \\
\left.+\left(r-\mu^{S}\right) \bar{e}\left(1-\kappa_{S}\right) \beta^{\prime \prime}(1-\mathcal{P}(L))\left(r+\lambda-2 \mu^{S}\right)\right)
\end{gathered}
$$

Observe that if $\beta^{\prime \prime}(z)=0$ for all $z \in[1-r, 1]$, then $(\lambda-r) \beta^{\prime}(1-\mathcal{P}(L))+r \beta^{\prime}(1-r)+\left(r-\mu^{S}\right) \bar{e}(1-$ $\left.\kappa_{S}\right) \beta^{\prime \prime}(1-\mathcal{P}(L))\left(r+\lambda-2 \mu^{S}\right)=\lambda \beta^{\prime}(1-r)<0$ so $S D<0$ and $\frac{d \bar{e}(L)}{d L}>0$. We now show that there exists a strictly positive lower bound on the second derivative such first-period performance
is strictly increasing with violence.
Suppose $\widehat{\beta}=\min _{z \in[1-r, 1]} \beta^{\prime \prime}(z)<0$ and note that $\beta^{\prime}(1-\mathcal{P}(L))-\beta^{\prime}(1-r)=\int_{1-r}^{1-\mathcal{P}(L)} \beta^{\prime \prime}(z) d z \geq$ $\widehat{\beta} \times(1-\mathcal{P}(L)-(1-r))=\widehat{\beta} \times\left(r-\mu^{S}\right)\left(1-\kappa_{S}\right) \bar{e}_{1}$. Assume $r+\lambda-2 \mu^{S}<0$ (a similar reasoning holds if the inequality is reversed), we hence have:

$$
\begin{aligned}
S D< & (1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e}))\left((\lambda-r) \beta^{\prime}(1-\mathcal{P}(L))+r \beta^{\prime}(1-r)\right. \\
& \left.+\left(r-\mu^{S}\right) \bar{e}\left(1-\kappa_{S}\right) \beta^{\prime \prime}(1-\mathcal{P}(L))\left(r+\lambda-2 \mu^{S}\right)\right) \\
< & (1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e}))\left(\lambda \beta^{\prime}(1-r)-2\left(r-\mu^{S}\right)\left(\mu^{S}-\lambda\right) \bar{e}_{1}\left(1-\kappa_{S}\right) \widehat{\beta}\right) \\
< & (1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e}))\left(\lambda \beta^{\prime}(0)-2\left(r-\mu^{S}\right)\left(\mu^{S}-\lambda\right) \widehat{\beta}\right)
\end{aligned}
$$

So for $\widehat{\beta} \geq \max _{L \in\left[L^{\text {ind }}, \bar{L}\right]}\left\{\frac{\lambda \beta^{\prime}(0)}{2\left(r-\mu^{S}\right)\left(\mu^{S}-\lambda\right)}\right\}, S D<0$ and $d \bar{e}_{1} / d L>0$. Hence, there exists $\underline{\beta}<0$ such that if $\min _{z \in[1-r, 1]} \beta^{\prime \prime}(z)>\underline{\beta}$, the fear effect holds for all $L \geq L^{\text {ind }}$.

Our last proposition establishes that the love effect can also be negative in this setting.
Proposition D.4. There exists $\underline{\underline{\beta}}<0$ such that if $r \geq \lambda, \epsilon<\bar{\epsilon}$ (with $\bar{\epsilon}$ defined in the text of Proposition D.1, and $\min _{z \in[1-r, 1]} \beta^{\prime \prime}(z)>\underline{\underline{\beta}}$, then:
(i) The proportion of congruent types among surviving subordinates of the purge strictly increases with $L$ if and only if $L<L^{\text {full }}$, and strictly decreases otherwise.
(ii) The proportion of congruent types among subordinates in the second period weakly increases with $L$ for all $L$ if and only if $\lambda \geq r$.
(iii) If $r \in(\lambda, 2 \lambda]$, the proportion of congruent types among subordinates in the second period strictly increases with $L$ for $L<L^{\text {full }}$ and strictly decreases otherwise.

Proof. Points (i) and (ii) follows directly from a similar reasoning as the proof of Proposition 4 since the purge inference is strictly increasing with $L$ under the assumption $\epsilon<\bar{\epsilon}$.
For point (iii), the claim holds directly for a fully discriminate purge (see Equation B.11). We thus focus on a semi-discriminate purge for which $\mathcal{P}(L)=r-\left(1-\kappa_{S}\right) \bar{e}\left(r-\mu^{S}\right)$ (ignoring subscripts, superscripts, and arguments). So as before $\frac{d \mathcal{P}(L)}{d L}=-\frac{d\left(1-\kappa_{S}\right) \bar{e}}{d L}\left(r-\mu^{S}\right)+\mu_{L}^{S}\left(1-\kappa_{S}\right) \bar{e}$ (with $\mu_{L}^{S}$ the partial derivative of $\mu^{S}$ with respect to $L$ ). From Equation D. 5 and $S_{S D}\left(\kappa_{S}^{*}(L), L\right)=0$, treating $\left(1-\kappa_{S}\right) \bar{e}$ as our variable of interest (and again ignoring subscripts, superscripts, and arguments
whenever possible), we obtain:

$$
\begin{aligned}
& (1-\gamma) \epsilon\left(1-\kappa_{S}\right)\left(r-\mu^{S}\right) \beta^{\prime}(1-\mathcal{P}(L)) \\
& +(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right) \mu_{L}^{S}\left(\beta^{\prime}(1-\mathcal{P}(L))+\left(1-\kappa_{S}\right) \bar{e}\left(r-\mu^{S}\right) \beta^{\prime \prime}(1-\mathcal{P}(L))\right) \\
& +\frac{d\left(1-\kappa_{S}\right) \bar{e}}{d L}\left(C_{1}-(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right)\left(r-\mu^{S}\right)^{2} \beta^{\prime \prime}(1-\mathcal{P}(L))\right)=0
\end{aligned}
$$

Hence, we obtain that $d \mathcal{P}(L) / d L$ has the same sign as:

$$
\begin{aligned}
\Upsilon:= & (1-\gamma) \epsilon\left(1-\kappa_{S}\right) \beta^{\prime}(1-\mathcal{P}(L))\left(r-\mu^{S}\right)^{2} \\
& +(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right) \mu_{L}^{S}\left(\beta^{\prime}(1-\mathcal{P}(L))+\left(1-\kappa_{S}\right) \bar{e}\left(r-\mu^{S}\right) \beta^{\prime \prime}(1-\mathcal{P}(L))\right)\left(r-\mu^{S}\right) \\
& +C_{1} \mu_{L}^{S} \bar{e}\left(1-\kappa_{S}\right)-(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right)\left(r-\mu^{S}\right)^{2} \beta^{\prime \prime}(1-\mathcal{P}(L)) \mu_{L}^{S} \bar{e}\left(1-\kappa_{S}\right) \\
= & (1-\gamma) \epsilon\left(1-\kappa_{S}\right) \beta^{\prime}(1-\mathcal{P}(L))\left(r-\mu^{S}\right)^{2} \\
& +\mu_{L}^{S}\left((1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right) \beta^{\prime}(1-\mathcal{P}(L))\left(r-\mu^{S}\right)+C_{1} \bar{e}\left(1-\kappa_{S}\right)\right)
\end{aligned}
$$

Given Equation D.5, we can rewrite the equality as:

$$
\begin{aligned}
\Upsilon= & (1-\gamma) \epsilon\left(1-\kappa_{S}\right) \beta^{\prime}(1-\mathcal{P}(L))\left(r-\mu^{S}\right)^{2} \\
& +\mu_{L}^{S}\left(2(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right) \beta^{\prime}(1-\mathcal{P}(L))\left(r-\mu^{S}\right)+C_{0}+C_{1}\right)
\end{aligned}
$$

Using $\mu_{L}^{S}<0$ and $C_{0}+C_{1}>-(1-\gamma)((1-\epsilon)+\epsilon(1-\bar{e})) r \beta^{\prime}(1-r)$, we obtain:

$$
\begin{aligned}
\Upsilon< & (1-\gamma) \epsilon\left(1-\kappa_{S}\right) \beta^{\prime}(1-\mathcal{P}(L))\left(r-\mu^{S}\right)^{2} \\
& +\mu_{L}^{S}(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right)\left(2 \beta^{\prime}(1-\mathcal{P}(L))\left(r-\mu^{S}\right)-r \beta^{\prime}(1-r)\right)
\end{aligned}
$$

Observe that if $\beta^{\prime \prime}(z)=0$ for all $z \in[1-r, 1], 2 \beta^{\prime}(1-\mathcal{P}(L))\left(r-\mu^{S}\right)-r \beta^{\prime}(1-r)=\beta^{\prime}(1-r)\left(r-2 \mu^{S}\right)>$ 0 under the assumption that $r \leq 2 \lambda$. This implies $\Upsilon<0$. By a similar reasoning as in the proof of Proposition D.3, there exists $\underset{\underline{\beta}}{\underline{\beta}} 0$ such that if $\min _{z \in[1-r, 1]} \beta^{\prime \prime}(z) \geq \underline{\underline{\beta}}$, then $\Upsilon<0$ and $d \mathcal{P}(L) / d L<0$ as claimed.

Since the fear and love effects are still present in this setting, A similar reasoning as in Appendix C yields conditions such that the purge is partially discriminate or semi-indiscriminate. There are, however, two important differences. First, we do not have a condition such that $\kappa_{F}^{*}(L)$ is continuous so Remark C. 1 applies. Second, without imposing additional conditions on $\beta(\cdot)$, we cannot determine whether the benefit of investing in the infrastructure of violence is convex for $L \leq L^{\text {full }}$ so Proposition 6 does not necessarily apply in this setting.

## D. 4 Repression

In this subsection, we suppose that the survival probability of the autocrat depends negatively on the mass of non-congruent subordinates in the party $\mathcal{N}(L)$. That is, the survival probability is:

$$
\begin{equation*}
P(\text { survives })=\gamma \bar{e}_{1}+(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right) \times \beta(\mathcal{N}(L)), \tag{D.10}
\end{equation*}
$$

The mass of non-congruent subordinates after the repression is:
(i) $\mathcal{N}(L)=\left(1-\bar{e}_{1}\right)\left(1-\kappa_{F}\right)\left(1-\mu^{F}\right)+\bar{e}_{1}\left(1-\mu^{S}\right)$ when repression is partially discriminate;
(ii) $\mathcal{N}(L)=\bar{e}_{1}\left(1-\mu^{S}\right)$ when repression is fully discriminate;
(iii) $\mathcal{N}(L)=\bar{e}_{1}\left(1-\kappa_{S}\right)\left(1-\mu^{S}\right)$ when repression is semi-indiscriminate.

Define

$$
\begin{equation*}
R_{P D}\left(\kappa_{F}, L\right)=-(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right)\left(1-\mu^{F}\right) \beta^{\prime}(\mathcal{N}(L))-C_{0}-C_{1} \kappa_{F}\left(1-\bar{e}_{1}\right) \tag{D.11}
\end{equation*}
$$

If the repression is partially discriminate, $\kappa_{F}^{*}(L)$ is defined as a solution to $R_{P D}\left(\kappa_{F}, L\right)=0$ since the autocrat takes effort and violence as given at the time of her purging decision.

Define

$$
\begin{equation*}
R_{S D}\left(\kappa_{S}, L\right)=-(1-\gamma)\left((1-\epsilon)+\epsilon\left(1-\bar{e}_{1}\right)\right)\left(1-\mu^{S}\right) \beta^{\prime}(\mathcal{N}(L))-C_{0}-C_{1}\left(1-\left(1-\kappa_{S}\right) \bar{e}_{1}\right) \tag{D.12}
\end{equation*}
$$

If the purge is partially discriminate, $\kappa_{S}^{*}(L)$ is defined as a solution to $R_{S D}\left(\kappa_{S}, L\right)=0$ since the autocrat takes effort and violence as given at the time of her purging decision.

Comparing Equation D.11 D. 12 and Equation D.4 D.5, it can be checked that we can apply a similar reasoning as in the previous section to show that as long as $\epsilon$ is sufficiently small:
(i) discriminate repression tends to be mild and semi-indiscriminate repression violent (Proposition D.1);
(ii) the size of repression is non-monotonic in violence (Proposition D.2);
(iii) the autocrat faces a trade-off between fear and love when choosing the intensity of violence if $\beta(\cdot)$ is not "too concave" (Propositions D. 3 and D.4).

## E Single agent set-up

In this last section, we study a model with a single agent rather than a mass of agents. As our goal is to illustrate the differences with our baseline model, we only perform a comparative statics on violence and do not consider the autocrat's problem of choosing the optimal intensity of violence.

Our single-agent model imposes a binary level of effort 0 or 1 . The reason is that with continuous effort, the purge is semi-indiscriminate only for a set of parameter values of measure 0 (details available upon request).

Consider a variant of our model with with three players: an autocrat $(A)$, a single incumbent subordinate $(I)$, and a potential new subordinate $(N)$. At the end of period 1 , the autocrat decides whether to purge the current subordinate $I(k \in\{0,1\}$, with $k=1$ denoting $I$ being purged). If agent $I$ is purged, then $N$ becomes the autocrat's subordinate. Further, if purged, agent $I$ suffers a loss $L \geq 0$. Each period, whomever is the autocrat's agent works on a project. A project can be a success $\omega=S$ or a failure $\omega=F$. The probability a project is successful depends on the agent's costly effort, which takes value $e \in\{0,1\}$. The cost of effort is $c(e)=\rho \times e$, with $c>0$ and the probability that the project is successful is $\operatorname{Pr}(\omega=S)=q \times e$, with $q \in(0,1)$ (and $q$ common knowledge).

The incumbent agent $I$ is either congruent $\left(\tau_{I}=c\right)$ or non-congruent $\left(\tau_{I}=n c\right)$. $I$ 's type is his private information. However, it is common knowledge that there is a probability $\lambda \in(0,1)$ that $I$ is congruent: $\operatorname{Pr}\left(\tau_{I}=c\right)=\lambda$. Similarly, $N$ is either congruent or non-congruent. His type is his private information and the probability that $N$ is congruent is $r: \operatorname{Pr}\left(\tau_{N}=c\right)=r \in(0,1)$. All types enjoy a payoff $R>0$ from being a regime insider. In addition, a type $\tau \in\{c, n c\}$ gets a payoff $v(F, \tau)=0$ from a non-successful project and $v(S, \tau)$, with $v(S, c)>0$ and $v(S, n c) \in[0, v(S, c))$ from a successful project. I's payoff in period 1 is

$$
\begin{equation*}
u_{1}^{I}(e ; \tau)=R+(1-k) \times v(\omega, \tau)+k(-L)-\rho \times e \tag{E.1}
\end{equation*}
$$

In the second period the payoff of subordinate $J \in\{I, N\}$ is:

$$
\begin{equation*}
u_{2}^{J}(e ; \tau)=R+v(\omega, \tau)-\rho \times e \tag{E.2}
\end{equation*}
$$

The autocrat cares about the success of the agent's project. She gets a payoff of 1 when the project is successful and 0 otherwise. In addition, the autocrat pays a cost $C_{1}>0$ when she purges the agent $S$ at the end of period 1 . Her utility function can thus be represented as:

$$
\begin{equation*}
U_{A}(\kappa)=\mathbb{I}_{\left\{\omega_{1}=S\right\}}+\mathbb{I}_{\left\{\omega_{2}=S\right\}}-C_{1} \times k \tag{E.3}
\end{equation*}
$$

To summarize, the timing of the game is:

## Period 1:

1. $I$ and $N$ privately observe their type $\tau \in\{c, n c\}$;
2. I decides whether to exert effort on his project: $e \in\{0,1\}$;
3. The autocrat $A$ observes $\omega_{1} \in\{S, F\}$. She decides whether to purge $I$;
4. First-period payoffs are realised;

## Period 2:

1. The subordinate ( $I$ if not purged, $N$ if purged) chooses effort level;
2. $\omega_{2}$ and second-period payoffs are realized, the game ends.

The equilibrium concept is Perfect Bayesian Equilibrium. Notice that the autocrat observes only the outcome of the project in period 1 (not I's effort) before deciding whether to purge $I$. We impose D1 equilibrium refinement to facilitate comparison with the baseline model.

Throughout, we use the same notation as in the baseline model. $V_{2}(\tau)$ denotes an agent's expected payoff in period 2 as a function of his type. The (ex-ante) average payoffs are denoted by $\bar{v}=\lambda v(S, c)+(1-\lambda) v(S, n c)$ and $\overline{V_{2}}=\lambda V_{2}(c)+(1-\lambda) V_{2}(n c)$.

The agent's strategy is a mapping from his type $\tau$ to an effort level $e \in\{0,1\}$ denoted with slight abuse of notation $e(\tau) \in\{0,1\}$. A mixed strategy is denoted $\alpha: \tau \rightarrow \Delta(\{0,1\})$. For the autocrat, her purging strategy is a mapping from outcome $\omega$ to a purge decision $k \in\{0,1\}$. In particular, we denote the probability $I$ is purged after outcome $\omega \in\{F, S\} \kappa_{\omega}$ (the equivalent of the purge inference in the baseline model). Finally, denote $\mu^{\omega}(\alpha(c), \alpha(n c))$ the autocrat's posterior that $I$ is congruent after observing $\omega \in\{F, S\}$ ) when she anticipates (correctly in equilibrium) that $I$ plays the tuple of strategies $(\alpha(c), \alpha(n c))$. Denote $\alpha_{F}=\lambda(1-\alpha(c) q)+(1-\lambda)(1-\alpha(n c) q)$ the probability $I$ fails and $\alpha_{S}=1-\alpha_{F}$, the probability it succeeds.

To make the problem interesting, we impose two assumptions on parameter values. First, we suppose that only congruent agents exert effort in period $2(q v(S, n c)-\rho<0<q v(S, c)-\rho)$. This implies that $V_{2}(c)=R+q v(S, c)-\rho$ and $V_{2}(n c)=R$. Further, the autocrat's gain from replacing a non-congruent type with a congruent type is $\mathcal{D}^{c, n c}:=q$. Using this result, we assume that the autocrat has some incentive to purge when her agent plays a separating strategy $C_{1}<$ $\left(r-\mu^{F}(1,0)\right) \mathcal{D}^{c, n c}$. Observe that absent the first condition, a purge does not occur in this set-up.

First, observe that Lemma 1 holds in this setting due to the D1 equilibrium refinement. Second, a no effort equilibrium does not exist because of the D1 refinement. Third, there is no equilibrium in which a congruent type randomizes between effort and no effort. If so, $\kappa_{S}=0$ (since success perfectly reveals congruence) and a congruent type's expected payoff from effort is $q v(S, c)-\rho+$ $q \kappa_{F}\left(V_{2}(c)+L\right)+\left(1-\kappa_{F}\right) V_{2}(c)+\kappa_{F}(-L)$. If he does not exert effort, his expected payoff is
$\left(1-\kappa_{F}\right) V_{2}(c)+\kappa_{F}(-L)$. Under our assumption that $q v(S, c)-\rho>0$, a congruent type is never indifferent.

We thus look for three types of equilibria:
(i) Partially discriminate purge in which a non-congruent type randomizes between effort and no effort;
(ii) Fully discriminate purge in which a non-congruent type plays a possibly degenerate mixed strategy;
(iii) Semi-indiscriminate purge in which a non-congruent $S$ randomizes between effort and no effort.

## Type (i) equilibrium.

The equilibrium features:
(a) $\kappa_{F}=\frac{\rho-q v(S, n c)}{q\left(V_{2}(n c)+L\right)}$ and $\kappa_{S}=0$;
(b) $\alpha(c)=1$ and $\alpha(n c)$ is the solution to $\left(r-\mu^{F}(1, \alpha(n c))\right) \mathcal{D}^{c, n c}=C_{1}$, with $\mu^{F}(1, \alpha(n c))=$ $\frac{\lambda(1-q)}{\lambda(1-q)+(1-\lambda)(1-q \alpha(n c))}$.
This equilibrium exists if and only if $(r-\lambda) \mathcal{D}^{c, n c}<C_{1}$ and $q L>\rho-q\left(v(S, c)+V_{2}(c)\right)$.
In this equilibrium, the ex-ante probability a subordinate is congruent in the second period is $\mathcal{P}(L)=\alpha_{F}\left(\kappa_{F} \times r+\left(1-\kappa_{F}\right) \times \mu^{F}(1, \alpha(n c))\right)+\alpha_{S} \mu^{S}(1, \alpha(n c))$. Since $\kappa_{F}$ strictly decrease with $L$ and other quantities do not depend on $L, \mathcal{P}^{\prime}(L)<0$.

## Type (ii) equilibrium.

The equilibrium a.e features:
(a) $\kappa_{F}=1$ and $\kappa_{S}=0$;
(b) $\alpha(c)=1$ and $\alpha(n c)=0$;

This equilibrium exists if and only if $q L<\rho-q\left(v(S, c)+V_{2}(c)\right)$.
To see this, suppose that $\alpha(n c)=1$, then $\mu^{F}(1,1)=\mu^{S}(1,1)=\lambda$ and the autocrat either always purges or never purges except if $(r-\lambda) \mathcal{D}^{c, n c}=C_{1}$ (a knife-edge condition). A contradiction with the assumed equilibrium type. Suppose $\alpha(n c) \in(0,1)$, then given $\kappa_{F}=1$, it must be that $q\left(V_{2}(n c)+L\right)+q v(S, n c)-\rho=0$ again a knife-edge condition. Hence, almost always, the equilibrium is as described above.

In this equilibrium, the ex-ante probability a subordinate is congruent in the second period is $\mathcal{P}(L)=\alpha_{F} r+\alpha_{S}$, with $\mathcal{P}^{\prime}(L)=0$.

## Type (iii) equilibrium.

The equilibrium features:
(a) $\kappa_{F}=1$ and $\kappa_{S}=1-\frac{\rho}{q\left(v(S, c)+V_{2}(c)+L\right)}$;
(b) $\alpha(c)=1$ and $\alpha(n c)$ is the solution to $\left(r-\mu^{S}(1, \alpha(n c)) \mathcal{D}^{c, n c}=C_{1}\right.$ with $\mu^{S}(1, \alpha(n c))=$ $\frac{\lambda q}{\lambda q+(1-\lambda) q \alpha(n c)}$.
The equilibrium exists if and only if: $(r-\lambda) \mathcal{D}^{c, n c}>C_{1}$ and $q L>\rho-q\left(v(S, c)+V_{2}(c)\right)$.
In this equilibrium, the ex-ante probability a subordinate is congruent in the second period is $\mathcal{P}(L)=\alpha_{F} r+\alpha_{S}\left(\kappa_{S} r+\left(1-\kappa_{S}\right) \mu^{S}(1, \alpha(n c))\right.$. Since $\kappa_{S}$ strictly increase with $L$ and other quantities do not depend on $L, \mathcal{P}^{\prime}(L)>0$.

Using the results above, we can observe major differences with our baseline model.

1. A purge is fully discriminate (type (ii) equilibrium) only if violence is low rather than intermediary like in the baseline model.
2. The nature of the purge does not depend on the intensity of violence unlike in the baseline model since for $q L>\rho-q\left(v(S, c)+V_{2}(c)\right)$, it is fully determined by the quality of the replacement pool.
3. Fixing the nature of the purge, the fear effect is null as effort does not depend on violence.
4. The love effect is negative in a partially discriminate purge (positive in the baseline model) and positive in a semi-indiscriminate purge (negative under the sufficient assumption $r \leq 2 \lambda$ in the baseline model).

These four major differences imply that the many-to-one accountability problem we study in the main text is fundamentally different than a one-to-one accountability problem. The latter cannot be used to approximate the former.


[^0]:    ${ }^{1}$ If $K_{P D}\left(\kappa_{F}, L\right)$ crosses 0 from below at some $\kappa_{F}^{\prime}$ then it must be that $K_{P D}\left(\kappa_{F}, L\right)>0$ for $\kappa_{F}>\kappa_{F}^{\prime}$ since the function is strictly convex. This contradicts $K_{P D}(1, L)<0$.

[^1]:    ${ }^{2}$ Further, when $B(L)$ is not continuous, there exists additional conditions such that small changes in parameter values can lead to discontinuous changes in the equilibrium intensity of violence, purge breadth, and effort. It would reinforce the result described in Proposition 6

[^2]:    ${ }^{3}$ Notice that we do not compute the sign of $\frac{d^{2} B(L)}{d L^{2}}$ for $L>L^{\text {ind }}$. Simulations suggest that the sign is ambiguous. However, it is not critical for our argument.

[^3]:    ${ }^{4}$ Notice that we assume that $C_{0}+C_{1}>\beta r \mathcal{D}^{c, n c}$, Equation 8, and Equation C. 13 hold for all $C_{1} \leq \overline{C_{1}}(\cdot)$. Otherwise, the condition can be appropriately rearranged.
    ${ }^{5}$ It is important to observe that for all $\left(\lambda, r, v(S, c), v(S, n c), C_{0}, \zeta_{0}\right) \in \mathcal{P}^{d}$, the condition described in the text of the proposition is knife-edge. However, the properties of $\mathcal{P}^{d}$ indicate that this knife edge condition can arise for a non-trivial set of parameter values.

[^4]:    ${ }^{6}$ Observe that if $L^{*}(0) \in\left(0, L^{\text {full }}\right)$, we can always choose $\chi_{1}$ large enough so that $\widehat{T} \in\left(0, L^{\text {full }}\right)$.

[^5]:    ${ }^{7}$ Observe that in this environment, we do not know whether $\kappa_{F}^{*}(L)$ is convex in $L$. The statement of the proposition and the proof do not exclude intervals $\left[L^{1}, L^{2}\right], L^{1}<L^{2}<L^{\text {full }}$ such that $d \bar{e}_{1} / d L>1$ for all $L \in\left[L^{1}, L^{2}\right]$.

